Real Analysis I - Final

NAME (in print):

Due in my office by noon Tuesday 12/11.

All written work must be legible, grammatically clear and complete, and logically put together. Explain everything you write and draw conclusions, if necessary in words, using complete sentences. Correct answers alone will not earn full credit.

1. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be periodic if there is a number \( p > 0 \) such that \( f(x + p) = f(x) \) for all \( x \in \mathbb{R} \). Show that a continuous periodic function on \( \mathbb{R} \) is bounded and uniformly continuous.

2. Let \( f \) be a function on \([a, b]\) with finitely many discontinuous points. Prove that \( f \) is Riemann integrable.

Consider now the function
\[
 f(x) = \begin{cases} 
 \cos(\tan(\frac{1}{x})) + \sin(\cot(\frac{1}{x})) & \text{wherever this is well defined,} \\
 1 & \text{elsewhere.} 
\end{cases}
\]

Show that \( f \) is Riemann integrable on any bounded interval \([a, b]\).

3. Let \( f : [0, 1] \to \mathbb{R} \) be a differentiable function such that \( f(0) = 0 \) and \( f'(x) \neq 0 \) for all \( x \in [0, 1] \).

a) Show that \( f \) is strictly monotone and therefore its inverse \( g = f^{-1} \) exists.

b) Assume that \( \lim_{x \to 0} f'(x) \) exists. Compute
\[
 \lim_{x \to 0} \frac{\int_0^{f(x)} g(t)dt}{\int_0^{f(0)} f(t)dt}.
\]

4. Let
\[
 h(x) = \begin{cases} 
 e^{-\frac{1}{x^2}} & x \neq 0 \\
 0 & x = 0. 
\end{cases}
\]

a) Show that \( \lim_{x \to 0} \frac{h(x)}{x^k} = 0 \) for any \( k \in \mathbb{R} \).

b) Show that \( h^{(n)}(0) \) exists for all integer \( n \). Compute them.

c) Write down the Taylor expansion of \( h \) at \( x_0 = 0 \). Prove that we cannot have
\[
 h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n,
\]

and therefore the Taylor remainder \( R_n(0, c) \) cannot go to 0 as \( n \to \infty \).