If $R$ is a commutative ring with 1, then zero divisors of $R$ cannot be units and vice versa.

If $x \in R$ is a unit, then $\exists x^{-1}$ such that $x^{-1}x = 1$

Suppose $xy = 0$. Then $x^{-1}xy = y$.

$\therefore y = 0$.

$\therefore x$ is not a zero-divisor.

Suppose $x \neq 0$ is a zero-divisor (i.e. $\exists z \neq 0$ and $\exists y$ such that $xy = 0$)

Then $\exists z' y = z \neq 0$.

Car: 1. Fields are integral domains.
2. Subrings of int. domains are integral domains.
3. Subrings of fields are int. domains ($\mathbb{Z} \subseteq \mathbb{Q}$)
Then suppose $R$ is c.m.r.u., $I$ an ideal of $R$

\[ \text{then} \]

1) $I$ is prime $\iff \frac{R}{I}$ is an int. domain

2) $I$ is maximal $\iff \frac{R}{I}$ is a field.

**Pf:** 1) $I$ is prime $\iff \forall a, b \in R \ ab \in I \Rightarrow \begin{cases} a \in I \\ \lor b \in I. \end{cases}$

$\iff \begin{cases} ab I = I \Rightarrow a I = I \lor b I = I \end{cases}$

$I$ is the zero of $\frac{R}{I}$

$\iff \begin{cases} a I b I = 0_{R/I} \Rightarrow a I = 0 \lor b I = 0 \end{cases}$

$\iff \frac{R}{I}$ is an int. domain

(a c.m.r.u. is an int. domain $\iff$ for $I$ prime)

2) Suppose $I$ is maximal. (want $\frac{R}{I}$ = field)

let $x + I \neq 0$ in $\frac{R}{I}$

If $x + I$ is not a unit $\langle x + I \rangle \neq \frac{R}{I}$

If $\langle x + I \rangle = \frac{R}{I}$, then $1 + I \in \langle x + I \rangle$

$\exists a \in R \ ax + I = 1 + I$
We have \( \pi : R \to R/I \)

\[ \varphi^* (\langle x+I \rangle) \]

\[ R \to R/I \]

\[ x+I \]

\[ \therefore I \text{ is not max in } R \]

(Ex. Write this up)

If \( \frac{R}{I} \) is a field, then any nontrivial ideal contains a unit, so \( \frac{R}{I} = \frac{k}{I} \)

If \( I \) is not maximal, \( \exists J \ni I \neq \emptyset \neq \frac{R}{I} \)

\( \pi (J) \) is a proper ideal of \( \frac{R}{I} \), so fill in the details (it's too hot now).

\[ \langle x \rangle \text{ in } k[x] \quad \text{(}k\text{-field)} \]

is maximal

Alt. Proof. Define \( \varepsilon : k[x] \to k \)

\[ p \mapsto p(0) \]

\( \varepsilon \) is a hom (trivial)

\[ \ker \varepsilon = \langle x \rangle \quad \varepsilon \text{ is clearly onto} \]
by 1st iso then \( \frac{k[x]}{\ker \phi} \cong \text{im } \phi \), i.e.

\[
\frac{k[x]}{\langle x \rangle} \cong k \not\langle \text{field} \rangle, \quad \therefore \langle x \rangle \text{ is max.}
\]

\[
\langle x \rangle \in \mathbb{Z}[x]
\]

Similarly to above \( \frac{\mathbb{Z}[x]}{\langle x \rangle} \cong \mathbb{Z} \text{ int. dom.} \), a field.

\[\therefore \langle x \rangle \text{ is prime, but not max.}\]

\[
\langle x \rangle \subsetneq \langle x, 2 \rangle \subsetneq \mathbb{Z}[x]
\]

\(\uparrow\) all poly's with even const. coeff.
Localization

Define on $R \times S$ an equiv:

- $(r, s) \sim (r', s')$ (think $\frac{r}{s} = \frac{r'}{s'}$)

Prove:

- Reflexive:
  - $rs' = r's$

- Symmetric:
  - $r's'' = r''s'$

- Transitive:
  - $r's'' = r''s's$
  - $rs's'' = r''s's$

If $R$ were not a domain, define $[r, s] \sim [r', s']$

$\exists s'' \in S \quad rs's'' = r's's$

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Def. $S^{-1}R = \frac{R \times S}{\sim}$

Ops:

- $[r, s] \cdot [r', s'] = [rr', ss']$
- $[r, s] + [r', s'] = [rs + r's, ss']$

Any $s \in S$ is a unit in $S^{-1}R$

$[1, s] [s, 1] = [s, s] \sim [1, 1]$
Special case: \( R \)-domain, \( P \)-prime ideal.
then \( S = R \setminus P \) is a multiplicatively closed set.
We write \( R_P = S^{-1}R \)

Thus \( R_P \) is a local ring, i.e., it has a unique maximal ideal.

Note: if \( R \) is a local ring with max ideal \( M \),
then \( U(R) = R \setminus M \).

Note: the converse is also true. If \( R \)
is a ring, \( I \)-an ideal and \( R \setminus I = U(R) \), then \( R \) is local with \( I = \text{max. ideal} \).

If \( R \) is a domain, then \( S_0 \) is a prime ideal
let \( S = R \setminus S_0 \), \( Q(R) = S^{-1}R \) is a field
and we have an injection \( R \hookrightarrow Q \) \( a \mapsto [a,1] \)