Universal properties & Constructions

Def: A universal property is one that contains words "exists! morphism making the diagram commute."  

Example: Vector spaces.
Suppose \( S \) is a basis for a vector space \( V \).
If \( W \) is a vector space and \( f \) is a function \( S \rightarrow W \), then \( \exists! \) linear map \( \varphi: V \rightarrow W \) st. \( \varphi |_S = f \), where \( i: S \rightarrow V \) is the usual inclusion.

\[
\begin{array}{ccc}
    S & \xrightarrow{f} & W \\
    i \circ & \cong & \exists! \varphi \\
    V & \xrightarrow{\varphi} & W
\end{array}
\]

making the diagram commute.

Example: Groups
Suppose \( G \) is a group and \( N \triangleleft G \).

\[
G \xrightarrow{p} G/N
\]

\( G/N \) is the group of cosets of \( G \).

If \( \varphi: G \rightarrow \overline{G} \) is a group homomorphism such that \( \varphi(N) \) is trivial, then
\[ \exists \psi : \frac{G}{N} \to \overline{G} \text{ s.t. } \psi p = \varphi \]

Define \( \psi \) as follows. Given \( aN \in \frac{G}{N} \)

\[ \psi(aN) = \varphi(a) \]

Well-defined: Suppose \( b \in aN \)

\[ \psi(bN) = \varphi(b) \]

Write \( b = ah \), where \( h \in N \)

\[ \varphi(b) = \varphi(ah) = \varphi(a) \cdot \varphi(h) = \varphi(a) \cdot e = \varphi(a) \]

\[ (\psi p)(a) = \psi(p(a)) = \psi(aN) = \varphi(a) \]
Example 1: If \( f: G \to H \) is a homomorphism of abelian groups, we define
\[
\text{Coker } f = \frac{H}{f(G)}.
\]
The natural projection \( p: H \to \frac{H}{f(G)} \) is universal among maps from \( H \) whose composition with \( f \) is 0.

In the category of real vector spaces, \( \ker f \) measures the degrees of freedom and \( \text{coker } f \) measures the constraints of the right-hand-side when solving the linear system \( f(g) = h \).

Clearly \( p \circ f = 0 \). Conversely, suppose \( q: H \to J \) is an abelian group hom., s.t. \( q \circ f = 0 \).

Define \( \Phi: \text{coker } f \to J \) by \( \Phi(h + f(G)) = q(h) \).

If \( h' + f(G) = h + f(G) \), \( h - h' \in f(G) \)

so \( q(h - h') = q(h) - q(h') = 0 \), so \( q(h) = q(h') \)

implies \( \Phi \) is well-defined.

\( \Phi(p(h)) = \Phi(h + f(G)) = q(h) \) \( \therefore \Phi \circ p = q \)

and our def. of \( \Phi \) is clearly unique to make this work.

\[
\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\downarrow{q} & & \downarrow{p} \\
\frac{H}{f(G)} & \xrightarrow{f} & J
\end{array}
\]

\( \exists \Phi \) s.t. \( q = \Phi \circ p \)
Universal constructions

Let \( C \) be a category.

Let \( D \) be a diagram in \( C \) (could be infinite).

\[ \cdots \xrightarrow{f} \]

\( \text{Lim}(D) \) is an object and a collection of morphisms to \( D \), that is universal among such things, i.e.

If \( W \) is an object in \( C \) with morphisms to \( D \) then \( \exists ! \varphi : W \to \text{Lim}(D) \) making the diagram commute.

\( \text{Colim}(D) \) is an object and morphisms from \( D \) that is universal (\( \omega \)-universal) among such things, i.e. If \( W \) is an object with morphisms from \( D \) then \( \exists ! \varphi : \text{Colim}(D) \to W \) making the diagram commute.
Example: Let $D$ be

then $\lim (D)$ is called product $\Pi$.

A $\text{colim} (D)$ is called co-product $\amalg$.

Universal constructions are unique up to iso.

$L \Rightarrow L$; identity works.

By uniqueness, composition of green maps $= \text{id}$.

Similarly for the other composition.
Sets  

\[ \text{Product} = \text{cartesian product} \]

\[ f_1 \quad \xrightarrow{f(W)} \quad f_2 \]

\[ \text{usual coordinate projections} \]

Suppose \( W \) is a set with functions \( f_1 : W \rightarrow A_1 \)
\( f_2 : W \rightarrow A_2 \)

Define: \( \varphi : W \rightarrow A_1 \times A_2 \) by
\[ \varphi(w) = [f_1(w), f_2(w)] \]

\[ (p_i \varphi)(w) = p_i(\varphi(w)) = p_i([f_1(w), f_2(w)]) \]
\[ = f_i(w) \]

(clearly unique)

Co-products  
Disjoint union, \( i_1 \)-inclusions.

\[ \text{Define } \varphi : A_1 \sqcup A_2 \rightarrow W \]
\[ \text{By } \varphi(a) = \begin{cases} g_1(a) & \text{if } a \in A_1 \\ g_2(a) & \text{if } a \in A_2 \end{cases} \]

\[ (\varphi \circ i_1)(a) = \varphi(i_1(a)) = \varphi(a) = g_1(a) \]
Another diagram (pullback)

\[ \lim \rightarrow \bullet B \]
\[ \downarrow \]
\[ A \rightarrow \bullet C \]

(the map to C is not drawn)

Product = subset of A \times B

of pairs \([a,b]\) s.t. a and b go to the same element in C.

Special case: \( A, B \in U \) and the arrows are inclusions

\[ \lim = A \cap B \]

Special case of colimit: \( A, B \in U \), \( C = A \cup B \)

\[ \text{colimit} = A \cup B \]

Top

Same stuff, but maps must be cont.

Pointed

\((A, a_0) \times (B, b_0) = (A \times B, [a_0, b_0])\)

Disjoint union \((A, a_0) \sqcup (B, b_0)\) with \(a_0\) and \(b_0\) identified.

\[ A \bigcirc a_0 \quad \bigcirc b_0 \]

(wedge product)

\[ A \lor B \]
Product = Cartesian product

Co-product = direct sum

\[ i_B(b) = \{0, b\} \]

\[ \varphi([a, b]) = g_A(a) + g_B(b) \]

Difference between product and sum:

\[
\prod_{i=1}^{\infty} G_i = \{ [g_1, g_2, \ldots] : g_i \in G_i \}
\]

\[
\bigoplus_{i=1}^{\infty} G_i = \{ [g_1, g_2, \ldots] : g_i \in G_i \text{ and all but finitely many } g_i \text{ are zero} \}
\]

Product = cartesian product

Co-product = wreath product

\[ G_1 \ast G_2 = \{ \text{all reduced words with alphabet = disjoint union of } G_1 \text{ and } G_2 \} \]

Reduced?

\[(a_1 b_1) a_2 \sim \]

\[ C_1 \]

Operation = catenation of words + reduce

(assoc., see Thatcher)
Van Kampen’s Theorem

Suppose \( X = \bigcup A_i \) \( A_i \) - open, path connected

\( \forall i,j \ A_i \cap A_j \) is path connected, then \( \varphi \) defined below is surjective.

Inclusions \( A_i \rightarrow X \) induce

\[ j_k : \pi_1(A_k) \rightarrow \pi_1(X) \]

By the universal property of \( \ast \)

\[ \exists ! \varphi : \ast \pi_1(A_k) \rightarrow \pi_1(X) \]

We also have inclusions \( A_k \cap A_c \rightarrow A_k \)

and they induce \( \iota_{ke} : \pi_1(A_k \cap A_c) \rightarrow \pi_1(A_k) \)

If \( \forall k, l, m \ A_k \cap A_c \cap A_m \) is path connected, then \( \ker \varphi = N = \langle \{ i_{ke}(\omega) \ i_{kl}(\omega)^{-1} : k, l \} \rangle \)

\[ (\text{so } \pi_1(X) \cong \ast \pi_1(A_i) / N) \]
Example

$S^1$

$A_1 \cap A_2 = \{1\}$

not path conn.

$\pi_1(A_1) = 0$, so $\pi_1(A_1) \ast \pi_1(A_2) = 0$

$\pi_1(S^1) = \mathbb{Z}$, so $\phi$ is not surjective.

Example

$X$

$\pi_1(A_i) = \mathbb{Z}$

$\prod_{i=1}^{3} \pi_1(A_i) = \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$

$\pi_1(X) = \mathbb{Z} \ast \mathbb{Z}$
Proof of van Kampen's Theorem (cf. Hatcher)

Breakup loop $\gamma$. For example with $2$ sets in the cover:

\[ \gamma = \gamma_1 \gamma_2 \sim (\tau_1 \gamma_1 \tau_1^{-1})(\tau_2 \gamma_2 \tau_2^{-1}) \]

loop in $A_1$

loop in $A_2$

$\pi_1(A_1) \ast \pi_1(A_2)$

In general $\gamma : I \to X = \bigcup_{i \in J} A_i$

\[ \forall t \in I \exists j_t \in J \quad \gamma(t) \in A_{j_t} \quad \text{i.e.} \quad t \in \gamma^{-1}(A_{j_t}) \]

so $\exists \delta_t > 0$ $(t - \delta_t, t + \delta_t) \cap I \subseteq \gamma^{-1}(A_{j_t})$

By slightly shrinking $\delta_t$ we can ensure

\[ [t - \delta_t, t + \delta_t] \cap I \subseteq \gamma^{-1}(A_{j_t}) \quad \text{(e.g. replace $\delta_t$ by $\delta_t/2$)} \]

Since $I = \gamma^{-1}(X) = \gamma^{-1}(\bigcup A_i) = \bigcup \gamma^{-1}(A_i)$ we have an open cover for $I$, subordinate to the cover of $X$ by $A_i$'s.

Since $I$ is compact, we can choose a finite subcover, i.e.

$\exists t_k \in I, \quad k = 1, \ldots, n$ s.t. $I = \bigcup_{k=1}^{n} \gamma^{-1}(A_{j_{t_k}})$

Then $\{ t_k + \delta_{t_k} : k = 1, \ldots, n \}$ gives a partition of $I$ subordinate to $A_i$'s.
A factorization $\delta_1, \delta_2, \ldots, \delta_n$ of $\delta$ in $X$ is homotopic to $\delta$ and $\forall i: \delta_i$ is in $A_i$.

$\therefore \varphi$ is onto.

Two factorizations are equivalent if you can get from one to the other via

1. Combining adjacent loops if they are in the same $A_i$

2. If a loop is in $A_i \cap A_j$, then we regard it as in $\pi_1(A_i)$ or in $\pi_1(A_j)$, so we permit changing between these. This is exactly $i_{ij}(w)i_{ji}(w)^{-1}$

The rest of the proof is to show any two factorizations are eq. ($\equiv \mod N$)

\[
\begin{array}{c}
\delta_1 \ldots \delta_n \\
\delta_1 \ldots \delta_n
\end{array}
\xrightarrow{\text{etc}}
\begin{array}{c}
\text{grid}
\end{array}
\]
Example 1. \( S^1 \subseteq \mathbb{R}^3 \)

Complement is homotopically equiv. to \( S^2 \vee S^1 \) (coproduct of pointed top. spaces)

\[ \pi_1(S^2 \vee S^1) = \pi_1(S^2) \ast \pi_1(S^1) = 0 \ast \mathbb{Z} = \mathbb{Z} \]

2. Two circles

\[ \pi_1 = \mathbb{Z} \ast \mathbb{Z} \]

3. Two linked circles

\[ T^2 \vee S^2 \]

\[ \pi_1 = (\mathbb{Z} \times \mathbb{Z}) \ast 0 = \mathbb{Z} \times \mathbb{Z} \]