Brouwer's Thm Results

Figure 1

String a

Strings

Figure 2

Figure 3
Today I am going to prove two things. The first will be a Lemma for Brouwer’s Fixed Point Theorem and the second will be Brouwer’s Fixed Point Theorem itself.

**Lemma** - $S^{n-1}$ is not a retract of $B^n$

**Proof**: This will be a proof by contradiction. Assume that $S^{n-1}$ is a retract of $B^n$. That means that there is a continuous function $f : B^n \to S^{n-1}$ so that $f|_{S^{n-1}}$ is the identity on $S^{n-1}$. Let $j : S^{n-1} \to B^n$ be the inclusion map. This means that $f \circ j : S^{n-1} \to S^{n-1}$ is the identity. From previous work in homology groups, we know that

$$H_{n-1}(S^{n-1}) = R$$
$$H_{n-1}(B^n) = \text{trivial}$$

and since functors are homomorphisms,

$$0 = (id)_*$$
$$= (f \circ j)_*$$
$$= f_* \circ j_*$$

which clearly implies that $f_* \circ j_* : \mathbb{R} \to 0 \to \mathbb{R}$. This is impossible.

Now I will use this lemma to prove Brouwer’s Fixed Point Theorem.

**Theorem** - (Brouwer’s Fixed Point Theorem) Every continuous function $f : B^n \to B^n$ has a fixed point $x \in B^n$ such that $f(x) = x$.

**Proof**: Again, we are going to use a contradiction to prove this Theorem. Let’s assume that $f : B^n \to B^n$ has no fixed point. Define $r : B^n \to S^{n-1}$ as the function that maps $x$ to through $f(x)$ to the closest point on $S^{n-1}$.
r would be considered a retraction, which, by the above Lemma, this would be impossible. Therefore, f has at least 1 fixed point.

Now let's take about some of the applications that Brouwer's Theorem. Some fun applications include:

- 1 - dimensional: Take a strip of string, then layer another string on top of it as such. By Brouwer's fixed point there there some point on the string that aligns perfectly with the point above it. Figure 1 pictorially describes this happening.

- 2 - dimensional: Grab two sheets of paper and label them one and two respectively. Crumple up paper two and place it anywhere over paper 1. Again, there will be one point on that sheet that will not have moved. Figure 2 pictorially describes this happening.

- 3 - dimensional: Take a coffee cup filled with coffee. Take that coffee cup and swish around the coffee inside the cup. When the coffee settles in the cup there will be 1 molecule that will be in the same place that it started. Figure 3 pictorially describes this happening.

- Nash - Equilibrium: Some more applications that are not so gimmicky include many existence questions since existence of solutions can be formulated as a fixed point problem.