Let $k$ be a field. Typically we'll assume $\text{alg. closed (think } \mathbb{C})$.

Let $n \in \mathbb{N}$, $k^n \subset \text{affine } n\text{-space}$

$$R = k[x_1, \ldots, x_n] \subset \text{polynomial ring in } n \text{ indeterminates.}$$

Given a subset $S \subset R$, define $Z(S) \subset \text{zero set}$

$$V(S) = \{ (a_1, \ldots, a_n) \in k^n : \forall p \in S \text{ } p(a_1, \ldots, a_n) = 0 \}$$

$\text{Variety (alg. set)}$

\[Z(S)\]

E.g. $V(\phi) = k^n$

$V(K) = \emptyset \iff n$

$n = 2 \quad V(\{ a_1 x_1 + a_2 x_2 - b \})$

\[\text{line } \rightarrow \]

$n = 3 \quad V(\{ a_1 x_1 + a_2 x_2 + a_3 x_3 - b \})$

\[\text{plane } \rightarrow \]

$$V(\{ a_1 x_1 + a_2 x_2 + a_3 x_3 - b \}, c_1 x_1 + c_2 x_2 + c_3 x_3 - d)$$

\[\text{line of intersection of two planes} \]
n=2 \quad V(\{x_1-a_1, x_2-a_2\})

Two points

V(\{(x_1-a_1)(x_1-b_1), x_2-a_2\})

Note: V is "decreasing"

Note: \(V(\{p_1, \ldots, p_m\}) = \bigcap_{i} V(p_i)\)

A commutative ring with unity \(R\)

is called Noetherian (a.k.a. has ACC, ascending chain condition)

if any monotone sequence of ideals terminates, i.e.

If \(I_1 \subseteq I_2 \subseteq I_3 \ldots\) are ideals, then \(\exists N \forall i, j \geq N I_i = I_j\).

Then any ideal in a Noetherian ring is f.g. (finitely generated)

If suppose \(I\) is not f.g.

let \(p_1 \in I\), then \(\langle p_1 \rangle \neq I\)

let \(p_2 \in I \setminus \langle p_1 \rangle\)

let \(p_3 \in I \setminus \langle p_1, p_2 \rangle\)

\((p_1, p_2) = p_1R + p_2R\)
\[(p_1, p_2) \subseteq (p_1, p_2, p_3) \subseteq \ldots \text{ does not terminate.}\]

Conversely, if we have a chain \( I_1 \subseteq I_2 \subseteq \ldots \) then \( U I_i \) is a ideal so f.g. \( : U I_i = \langle p_1, \ldots, p_m \rangle \) (since monotonous union).

\[\forall j \exists i_j \in I_{i_j} \text{ let } J = \max \{ i_j \}_{j=1}^m\]

so \( U_i I_i \) contains all \( p_i \) \( : U I_i = U_i I_i \) so the chain terminates at \( J \).

Hilbert Basis theorem: If \( R \) is noeth. so \( \bar{u} \in R[x] \)