Theorem. Suppose \( U \) is a set in \( \mathbb{R}^2 \)

\[ U \text{ irreducible } \iff \mathcal{I}(U) \text{ is a prime ideal.} \]

\[ \mathcal{I}(U) = \frac{k[x_1, \ldots, x_n]}{\mathcal{I}(U)} \text{ in an int. domain} \]

If \( R \) is a commutative ring with unity and \( \mathcal{I} \) is an ideal

then \( \frac{R}{\mathcal{I}} \) is a field \( \iff \mathcal{I} \) is max

and \( \frac{R}{\mathcal{I}} \) is an int. domain \( \iff \mathcal{I} \) is prime.

An ideal \( \mathcal{I} \) is prime means:

\[ ab \in \mathcal{I} \iff a \in \mathcal{I} \text{ or } b \in \mathcal{I} \]

\[ a \notin \mathcal{I}, b \notin \mathcal{I} \Rightarrow ab \notin \mathcal{I} \]

i.e. \( R \setminus \mathcal{I} \) is closed under multiplication.
Pf: Suppose $U$ is an irreducible algebraic set.

and $p, q \in k[x_1, \ldots, x_n]$, $pq \in \mathcal{I}(U)$

$U \subseteq V(pq) = V(p) \cup V(q)$

$U = (V(p) \cap U) \cup (V(q) \cap U)$

One of these $\uparrow$ say $V(p) \cap U = U$

Then $p \in \mathcal{I}(U)$.

Now assume $\mathcal{I}(U)$ is prime and

$U = U_1 \cup U_2$, $U_i \neq U$

$\Rightarrow$ alg. since

$U_i \not\subseteq U \Rightarrow \mathcal{I}(U) \not\subseteq \mathcal{I}(U_i)$

Pick $p_i \in \mathcal{I}(U_i) \setminus \mathcal{I}(U)$

$p_1p_2$ vanishes on $U \Rightarrow p_1p_2 \in \mathcal{I}(U)$.

\[ NAF: \Gamma(k^n) = \frac{k[x_1, \ldots, x_n]}{\mathcal{I}(k^n)} = k[x_1, \ldots, x_n] - \text{int. domain} \]

so $k^n$ is irreducible.

Density argument: if $p \in \mathcal{I}$ (nonempty Zariski open set)

then $p = 0$. 


Let \( U \) be an algebraic set \( \neq k^n \).

Suppose \( p \) vanishes in \( k^n \setminus U \).

\[ k^n \setminus U \subseteq \nu(p) \]

\[ k^n = (k^n \setminus U) \cup U \subseteq \nu(p) \cup U \]

\[ k^n = \nu(p) \cup U \cap \text{irred.} \]

Given a topological space \( X \) and a subset \( Y \), we topologize \( Y \) by saying \( U \subseteq Y \) is open whenever \( \exists U' \text{ open } \subseteq X \) \( U = U' \cap Y \).

This called relative top.

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Given \( Y \subseteq X \), closure \( \overline{Y} = \bigcap \{ Z \subseteq X \mid Z \text{ closed } \} \)

Note: \( \overline{Y} \) is closed

\[ \overline{Y} \text{ is closed} \]

Theorem: \( Y \) irreducible \( \Rightarrow \overline{Y} \) is irreducible.

If \( Y \) irreducible \( \Rightarrow \overline{Y} \) is irreducible.

Suppose \( \overline{Y} = U_1 \cup U_2 \)

\( \subseteq \nabla \text{alg. set (closed in } \overline{Y} \) \)

\( \Rightarrow \text{ closed in } X \)
Lemma: If \( A \subseteq B \) and \( B \subseteq C \),

then \( A \) is closed in \( C \).

\[ \text{If } B \setminus A \subseteq B \]
\[ \exists \; \exists u \subseteq C \quad u \cap B = B \setminus A \]

WTS: \( (C \setminus u) \cap B = A \)

\[ (C \setminus u) \cap B = (C \cap B) - (u \cap B) \]
\[ = B - (B - A) \]
\[ = A \]

\[ \overline{Y} = \cup_i u_i \cup u_2 \]
\[ Y = (u_1 \cap Y) \cup (u_2 \cap Y) \]

Y irredu. \( \Rightarrow \) \( \exists j \quad Y = U_j \cap Y \)

\( Y \subseteq U_j \subseteq \text{closed in } X \)

\[ \therefore \quad \overline{Y} = \bigcap Z \subseteq U_j \ni \text{one of the } Z's! \]

\[ \overline{Y} \subseteq Z \quad Z \subseteq X \quad Y \subseteq Z \]

\[ \therefore \quad \overline{Y} \subseteq U_j \]
\[ \therefore \quad \overline{Y} = U_j \cup \]
Given an alg. set \( U \neq \emptyset \in k^n \)

Thus \( U = U_1 U_2 \ldots U_m \) where \( U_i \) are irreducible alg. sets essentially uniquely.

Existence: Let \( S = \{ \text{alg. set : existence fails} \} \)

Let \( \mathcal{I} = \{ \Sigma(U) : U \in S \} \)

Since \( \mathcal{B} = \{ k_1 \ldots k_n \} \) is Noetherian, \( \mathcal{I} \) has a maximal element \( \Sigma(W) \)

Pick an ideal \( J_1 \in \mathcal{I} \)

- If \( J_1 \) is not maximal, then \( \exists J_2 \in \mathcal{I} \)

\[ J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq J_4 \]

\( \vdots \)  

non-terminating chain \( \hat{\circ} \)

If \( W \) is irreducible, done.

So assume \( W = W_1 U W_2 \), \( W_1 \subset W \), \( W_2 \subset \neq W \)

\( W_1 \subset W \implies \Sigma(W) \subset \Sigma(W_i) \) (maximality)

\( W_i \notin \mathcal{I} \)

\( W_1 \notin \mathcal{I} \)

\( W_2 \notin \mathcal{I} \)

\( W_2 = V_1 \ldots V_{n_2} \)
\[ W = W_1 \cup \ldots \cup W_k = u_1 \cup \ldots \cup u_r \cup v_1 \cup \ldots \cup v_k \]

\[ \iff W \in \mathcal{I} \]

\[ u = u_1 \cup \ldots \cup u_r \]
\[ = W_1 \cup \ldots \cup W_k \]

\[ u_1 = u \cap u_1 = (W_1 \cap u_1) \cup \ldots \cup (W_k \cap u_1) \]

Since \( u_1 \) is irreducible, \( \exists j \) \( u_1 = W_j \cap u_1 \)

\[ u_1 \subseteq W_j \subseteq u_1 \]

\[ u_1 \subseteq u_j \Rightarrow i = 1 \]

\[ u_1 \subseteq W_j \subseteq u_1 \]

\[ \therefore u_1 = W_j \]

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If \( W \subseteq U \) (alg. size) \( U = u_1 \cup \ldots \cup u_r \)

and \( W \) is irreducible.

\[ W = (u_1 \cap W) \cup \ldots \cup (u_r \cap W) \]

\[ \exists i \ W = u_i \cap W \]

\[ W \subseteq u_i \]

\[ \therefore \]
Localization

Suppose $R$ is an integral domain.

$A \subseteq R$ that is closed under multiplication.

Let $S = \{ [x, a] : x \in R, a \in A \}$

Define $[x, a] \sim [x', a']$

whenever $xa' = ax'$

Think:

$\frac{x}{a} = \frac{x'}{a'}$

Ex.

$\mathbb{Q}[\sqrt{2}]$ is an integral domain

with operations

$[x, a][x', a'] = [xx', aa']$

$[x, a] - [x', a'] = [xa' - ax', aa']$

Injection $R \rightarrow \frac{S}{\sim} = A^{-1}R$

$x \mapsto [ax, a]$ for some $a \in A$.

(well defined)
Exp. \( A = R \setminus I \) if \( I \) is a prime ideal.

If \( A = R \setminus \{0\} \), then the field of quotients

\( (R \text{ int. dom } = \Rightarrow \{0\} \text{ is a prime ideal}) \)

If \( A \subseteq R^* \), then you just get \( R \) back.

Let \( x \in R \), let \( A = \{1, x, x^2, x^3, \ldots \} \)

\( x \neq 0 \)