2. Let $V$ be an affine algebraic set in $\mathbb{K}^n$, and consider $x \not\in V$. Show that there is an $F \in \mathbb{K}[x_1, \ldots, x_n]$ such that $F(x) = 0$ and $F|_V = 0$.

$$V \subseteq \bigcup_{x \in \mathbb{K}^n} \{ x \} \quad \text{is closed, so } \bigcup_{x \in \mathbb{K}^n} \{ x \} \text{ is closed}$$

$$\mathcal{I}(\bigcup_{x \in \mathbb{K}^n} \{ x \}) \neq \mathcal{I}(V)$$

In particular, there exists an $F \in \mathcal{I}(V) - \mathcal{I}(\bigcup_{x \in \mathbb{K}^n} \{ x \})$.

Then $F|_V = 0$ and $F|_{\bigcup_{x \in \mathbb{K}^n} \{ x \}} \neq 0$. So

$$F(x) \neq 0.$$  Let $\overline{f} = \frac{F}{F(x)}$.  \(\Box\)

3. Let $F \in \mathbb{K}[X,Y]$ be an irreducible polynomial. Assume $V(F)$ is infinite. Prove that

$$\mathcal{I}(V(F)) = (F).$$

Clearly $F \in \mathcal{I}(V(F))$, so $(F) \subseteq \mathcal{I}(V(F))$. Fix $g \in \mathcal{I}(V(F))$. Then

$$V((f,g)) = V(f) \cap V(g) = V(f)$$ since $V(g) \neq V(f)$.  \(\Box\)
\[ V(g) = V(f) \cup F_i \cup \cdots \cup F_m \]

where \( F_i \) are closed sets and irreducible.

To be continued...

4. (a) If \( X \) is irreducible and \( U \) is an open subset of \( X \), show that \( U \) is irreducible.

Proof:

Let \( X \) be irreducible and assume, to the contrary, that \( U \) is not irreducible. Let \( V_1, V_2 \subset U \) be nonempty open subsets of \( U \) such that \( V_1 \cap V_2 = \emptyset \). Then \( V_1, V_2 \) open \( \subset X \), with \( V_1 \cap V_2 = \emptyset \), and since \( X \) is irreducible, \( V_1 = \emptyset \) or \( V_2 = \emptyset \).

So, \( U \) is irreducible.
Let \( W \) be an algebraically closed field. Then \( I(W) \) is maximal among the ideals of \( \Gamma(W) \).

\( \Leftrightarrow \Gamma(W) \) is a field.

Actually, \( \Gamma(W) = k \)

(see proof of Nullstellensatz)

Suppose \( W = \{ \bar{a} \} \)

and \( I(W) \not\trianglelefteq J \)

\[ W = V(I(W)) \supsetneq V(J) \]

\[ \therefore V(J) = \emptyset \]

\[ \sqrt{J} = I(V(J)) = I(\emptyset) = k[x_1 \ldots x_n] \]

\[ \therefore 1 \in \sqrt{J} \Rightarrow 1 \in J \Rightarrow J = k[x_1 \ldots x_n] \]

Suppose \( W \) is not a point.

If \( W = \emptyset \), then \( I(W) = k[x_1 \ldots x_n] \)

\[ \therefore I(W) \) is not maximal.

If \( W \neq \emptyset \), \( \exists \bar{a} \in W \)

\( \{ \bar{a} \} \not\subseteq W \) (since \( W \) is not a pt.)

\[ I(\{ \bar{a} \}) \supsetneq I(W) \not\trianglelefteq \text{maximal} \]

\( \text{maximal ideal } \langle x_1 - a_1, \ldots, x_n - a_n \rangle \)
**Theorem**: \( \dim U \text{ finite } \iff \Gamma(U) \text{ is f.d. over } \mathbb{k} \).

**Proof**: Let \( U = \{ \bar{u}_1, \ldots, \bar{u}_n \} \subset \mathbb{k}^n \).

Given \( p \in \mathbb{k}[x_1, \ldots, x_n] \),

define \( \varphi(p) \in \mathbb{k}^n \) by \( \varphi(p)_i = p(\bar{u}_i) \).

\( \ker \varphi = \{ p : p(\bar{u}_i) = 0 \text{ for all } i \} = I(U) \)

\( \Gamma(U) = \frac{\mathbb{k}[x_1, \ldots, x_n]}{I(U)} = \frac{\mathbb{k}[x_1, \ldots, x_n]}{\ker \varphi} \cong \text{Im}(\varphi) \subset \mathbb{k}^n \).

Suppose \( \Gamma(U) \text{ is f.d.} \)

\[ \mathbb{k}[x_1, \ldots, x_n] \rightarrow \frac{\mathbb{k}[x_1, \ldots, x_n]}{I(U)} = \Gamma(U) \]

\[ x_i \rightarrow \bar{x}_i \]

\[ \{ 1, \bar{x}_1, \bar{x}_2^2, \ldots \} \subset \Gamma(U) \implies \text{f.d.} \]

\[ \text{linearly dep. } \implies \exists \text{ nontriv. relation} \]

\[ a_0 + a_1 \bar{x}_1 + \cdots + a_s \bar{x}_i^s = 0 \]

\[ \bar{x}_i(\bar{u}) = u_i \quad (\text{for } \bar{u} \in U) \]
\[ a_0 + a_1 u_1 + \ldots + a_5 u_5^5 = 0 \]

works for only finitely many \( u_1 \)’s.

\[ \text{only finitely many } u_1 \text{'s.} \]