If $U$ is alg. set and $p \in \mathcal{F}(U)$

define the standard open set $U \setminus \mathcal{V}(p)$

Any open subset of $U$ is a union of standard open sets

Pf. Assume $k$ is alg. closed

pick an open set $D \subseteq U$, then $U \setminus D$ is closed in $U$.

But $U$ is closed in $K^n$, so $U \setminus D$ is closed in $K^n$.

$I(U \setminus D) = \langle \rho_1, \ldots, \rho_m \rangle$ by Hilbert Basis Theorem
\[ V(I \cup D) = U \setminus D = V_q \cup \ldots \cup V_{q_m} \]

\[ D = (U \setminus V_q) \cup \ldots \cup (U \setminus V_{q_m}) \]

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If \( X \) is of the form \( U \cup U_x \), where \( U_i \) is open and irreducible, \( U \cap U_x \neq \emptyset \), show that \( X \) is irreducible.

PS: Fix \( X_1, X_2 \) open in \( X \) such that \( X_1 \cap X_2 = \emptyset \). We wish to show that \( X_1 = \emptyset \) or \( X_2 = \emptyset \).

\[ X_1 = (X_1 \cap U_1) \cup (X_1 \cap U_2) \]
\[ X_2 = (X_2 \cap U_1) \cup (X_2 \cap U_2) \]

If \( X_1, X_2 \neq \emptyset \), then
\[ X_1 \cap X_2 \cap U_i \neq \emptyset \] and
\[ X_1 \cap X_2 \cap U_j \neq \emptyset \] for some \( i, j \).
If $i = \hat{j}$, and since $U_i$ is irreducible,

$X_i \cap U_i = \emptyset$ or $X_j \cap U_i = \emptyset$.

Without loss of generality, assume

$X_i \cap U_i$ and $X_j \cap U_i$ are nonempty.

We can assume $X_i \cap U_i = \emptyset$ and $X_j \cap U_i = \emptyset$.

$X_i \cap U_i$ and $U_i \cap U_j$ are open and nonempty in $U_i$.
Projective space (cont.)

Quotient topology: If \( \mathbb{X} \) is a topological space and \( \sim \) is an equiv. rel. on \( \mathbb{X} \).

Then \( \frac{\mathbb{X}}{\sim} = \{ [x] : x \in \mathbb{X} \} \)

Equiv. class \( \{ x' \in \mathbb{X} : x \sim x' \} \)

\( U \subseteq \frac{\mathbb{X}}{\sim} \) is open \( \iff \) \( U \ [x] \) is open in \( \mathbb{X} \).

Open covers give a basis for a topology on \( \mathbb{P}_n \).

We have a cont. map from \( S^n \to \mathbb{P}_n \) of degree 2.

Cor.: \( \mathbb{P}_n \) is compact & connected.

Notation: A pt. in \( \mathbb{P}_n \) is an equiv. class \( \bigg/ \sim \) of some pt. in \( \mathbb{R}^{n+1} \setminus \{0\} \).

Pick a representative \( \bar{x} = [x_0, \ldots, x_n] \) homogeneous coordinates.
Given a pt. with hom. coords \([x_0, \ldots, x_n]\)
and \(x \in \mathbb{R}^{n+1}\), \([2x_0, \ldots, 2x_n]\) is also
a set of homog. coords. for the same pt.
in \(\mathbb{P}^n\).

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Given a vecor subspace \(A \subset \mathbb{R}^{n+1}\)
if \(\overline{x} \neq 0\), \(\overline{x} \in A\), then \(A\) consists
\(\lambda \overline{x} \in A\) and conversely
if \(\overline{x} \in A\), \(\overline{x} \neq 0\), then \(A\) consists
\(\lambda \overline{x} \in A\).

\[\therefore A \text{ "respects" } \sim\]

So \(\frac{A}{\sim}\) is called a projective
subspace of \(\frac{\mathbb{R}^{n+1}}{\sim} = \mathbb{P}^n\)

projective line
$GL_{n+1}(k)$ acts on $\mathbb{P}^n$

\[ GL_{n+1}(k) = \{ \text{linear bijections of } \mathbb{F}^{n+1} \} \]

If we pick a basis for $\mathbb{F}^{n+1}$, then a linear bijection is represented by

$(n+1) \times (n+1)$ matrix

whose columns are the images of $e_0, \ldots, e_n$ in the standard basis:

\[ e_0 = [1, 0, \ldots] \]

with nonzero determinant.

Linear transformations take lines through $0$ to lines through $0$.

So a well-defined on $\mathbb{P}^n$:

Suppose $x, \tilde{x} \in \mathbb{F}^{n+1}$ with $y = \lambda x$ for some $\lambda \in \mathbb{F}$ and $\varphi \in GL_{n+1}(k)$, then

$\varphi(\tilde{x}) = \varphi(\lambda x) = \lambda \varphi(x)$

$GL_{n+1}(k)$ acts on $\mathbb{P}^n$:

(an element of $GL_{n+1}$ gives a homography of $\mathbb{P}^n$)
\[ \text{Exp } n \left[ \begin{array}{c} 2 \\ 0 \\ 0 \end{array} \right] \text{ act trivially} \]

2) A rotation \( \Omega h^{n+1} \) gives a shift in \( \mathbb{P}^n \).

Suppose \( H \subset h^{n+1} \) is a hyperplane.

For convenience pick \( H \) to be \( x_0 = 0 \).

If \( \bar{x} \in h^{n+1} \setminus H \), \( \bar{x} = [x_0 : x_1 : \ldots : x_n] \)

\[ \sim [1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] \]

\[ \mathbb{P}^n \setminus \bar{H} \cong \mathbb{R}^n \]

\[ \uparrow \text{ hyperplane at infinite} \]

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Given \( \varphi \in \text{GL}_2(\mathbb{R}) \)

\[ \varphi = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \]

\[ \varphi([x_0, x_1]) = \left[ \begin{array}{c} a \\ c \end{array} \right] \left[ \begin{array}{c} x_0 \\ x_1 \end{array} \right] = \left[ \begin{array}{c} ax_0 + bx_1 \\ cx_0 + dx_1 \end{array} \right] \]

\[ [x_0, x_1] \sim [1, \frac{x_1}{x_0}] \]

\[ \uparrow \]

\( \varphi \) gives \[ [a + bx_0, c + dx_0] \]

\[ \sim [1, \frac{c + dx_0}{a + bx_0}] \]
Homography \( n \in \mathbb{R}^2 \): \( x \mapsto \frac{c+dx}{a+bx} \)

\[\alpha_0 \mapsto \frac{1}{b}\]

\[-\frac{a}{b} \mapsto \infty\]

Fractional linear transf.
(Möbius transf.)

PL (projective linear group)