Sheaf theory 101

Let $X$ be a top. sp., regarded as a category with objects $U \in X$, arrows $U \rightarrow V$ for $U, V \in X$. Elements of $F(U)$ are called sections of $F$ over $U$. The collection of sections of $F$ over $U$ is denoted $\Gamma(U, F)$.

A presheaf on $X$ with values in $C$ is a contravariant functor $F : X \rightarrow C$. A morphism $\alpha : F \rightarrow G$ of presheaves is a natural transformation $\alpha : F \rightarrow G$.

A $C$-valued sheaf on $X$ is a $C$-valued presheaf $F$ on $X$ with unique gluing:
- If $U$ is covered by open $U_i$ and $F_i : F(U_i)$ is a collection of sections s.t. $p_{ij}(F_i) = p_{ji}(F_i)$,
- there is a unique $f \in F(U) \cap F(U_j)$ with $p_i(f) = F_i$.

Remark: later, *top. sp.* is too strong. There is a more general notion, called a Grothendieck topology, that goes enough to talk about sheaves. When you see they note that in a top. sp. $U_i \times X U_j = U_i \cap U_j$, and ideas of “covering” and “fibrational product” are enough to do what we need.

Examples: $R$, $C$, or otherwise-valued contravariant functions, stalks of such, orientations of manifolds, $C^k$ and analytic maps, meromorphic maps, poly. and rational maps on our varieties...

In particular, a ringed space is a space with a sheaf of rings, denoted by $O_X$ and called the structure sheaf of $X$. 

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Given ringed spaces $(X, O_X)$, $(Y, O_Y)$, is a continuous map $\gamma : X \to Y$ that takes 'nice functions' on $Y$ to 'nice functions' on $X$, i.e., composition takes $g \in \Gamma(U, O_Y)$ to $g \circ \gamma \in \Gamma(U', O_X)$.

Remark: a sheaf can be defined over a basis for our topology.

Let $V$ be an affine algebra, let $f \in \Gamma(V)$. Set

$$\Gamma(\text{Spec} f, O_V) = \Gamma(V)_f.$$  

The sheaf so obtained is called the sheaf of regular functions on $V$.

Sheaf axioms: exercise

An affine algebraic variety is an affine algebraic set equipped with a ring of functions, as above—more precisely, a ringed space (so, to a pair $(V, O_V)$).

A morphism of affine algebraic varieties is a morphism of ringed spaces that plays nicely with stalks (to be continued...).