Singularity bounds

Given a curve $V(F(x,y))$, $\deg F = n \geq 2$
how many double pts (counted properly)?

$F$ irreducible $\implies \# \text{ double pts} \leq \frac{(n-1)(n-2)}{2}$

($F$ is square-free $\implies \# \text{ double pts} \leq \frac{n(n-1)}{2}$)

Genus $g$ how badly this bound is off:

$g = \frac{(n-1)(n-2)}{2} - \# \text{ double pts}$. \hspace{1cm} (g \geq 0)

Cubic: $g = 1 - \# \text{ double pts}$, so

nonsingular cubics have $g = 1$ and
singular cubics have $g = 0$.

Facts: $g = 0 \iff$ the curve can be rationally parametrized.

$g$ is a birational invariant.
Exp unit circle: \( V(x^2 + y^2 = 1) \), \( g = 0 \).

Non-rational traditional parametrization: \( x = \cos \theta, y = \sin \theta \)

Rational parametrization

Let \( t = \tan \frac{\theta}{2} \)

Then

\[
X = \frac{1 - t^2}{1 + t^2} \quad Y = \frac{2t}{1 + t^2}
\]

\((\cos t)^2 + (\sin t)^2 = 1\)
\((\cos t)^2 - (\sin t)^2 = \cos (2t)\)

\((\cos t)^2 = \frac{1 + \cos (2t)}{2}\)  \(\left(\cos \frac{\theta}{2}\right)^2 = \frac{1 + \cos \theta}{2}\)

\((\sin t)^2 = \frac{1 - \cos (2t)}{2}\)  \(\left(\sin \frac{\theta}{2}\right)^2 = \frac{1 - \cos \theta}{2}\)

\((\tan \frac{\theta}{2})^2 = \frac{1 - \cos \theta}{1 + \cos \theta}\)

\((\tan \frac{\theta}{2})^2 + \cos \theta \left(\tan \frac{\theta}{2}\right)^2 = 1 - \cos \theta\)

\((1 + \cos \theta) \left(\tan \frac{\theta}{2}\right)^2 = 1 - \cos \theta\)

\(\cos \theta \left(\left(\tan \frac{\theta}{2}\right)^2 + 1\right) = 1 - \left(\tan \frac{\theta}{2}\right)^2\)

\[
X = \cos \theta = \frac{1 - \left(\tan \frac{\theta}{2}\right)^2}{1 + \left(\tan \frac{\theta}{2}\right)^2} = \frac{1 - t^2}{1 + t^2}
\]

\[
Y = \sqrt{1 - X^2} = \sqrt{1 - \frac{(1 - t^2)^2}{(1 + t^2)^2}} = \frac{\sqrt{(1 + t^2)^2 - (1 - t^2)^2}}{1 + t^2}
\]

\[
= \sqrt{\frac{(1 + t^2)(1 - t^2)}{(1 + t^2)^2}} = \frac{2t}{1 + t^2}
\]
Example: Consider the birational transformation

\[ x' = x, \quad y' = \frac{y}{x} \] (inverse \( x = x', \quad y = y'x' \))

Nodal cubic \( V(y^2 - x^2 - x^3) \) (\( g = 0 \))

\[ y'^2 x' - x'^2 - x'^3 \]

\[ x'^2 \left( y'^2 - 1 - x' \right) \]

\[ \text{parabola} \]

\[ \therefore \text{Nodal cubic is birationally equivalent to a parabola} \] (\( g = 0 \))

\[ x' = 0 \implies x = 0 \implies y' \text{ is not well-defined} \]

So the birational is defined on the nodal cubic w/o the origin. At other pts \( x' \neq 0 \)

So you just get the parabola.

Cuspidal cubic \( V(y^2 - x^3) \)

\[ y'^2 x'^2 - x'^3 = x'^2 \left( y'^2 - x' \right) \]

\[ \text{parabola} \] (\( g = 0 \))

Since the 2 parabolas are birationally equiv (shift \( x \) by \( 1 \))

Nodal & cuspidal cubics are birationally equivalent.

Fact: If \( \omega \) is a differential on the curve, then
\[ \# \text{zeros}(\omega) - \# \text{poles}(\omega) = 2g-2 \] (\( g \) is birationally invariant)