Please show all work.

1. Use induction to show that for \( n \geq 1 \) the partial sum
\[
1^3 + 2^3 + \ldots + n^3 = \sum_{k=1}^{n} k^3
\]
can be expressed in closed form by
\[
\left[ \frac{n(n + 1)}{2} \right]^2
\]

2. Use Euclid’s algorithm to find \((48, 22)\) and \(s, t \in \mathbb{Z}\) such that \((48, 22) = 48s + 22t\).

3. Compute \(3^{21}\) modulo 9 by repeated squaring and reduction. Show work.

4. Suppose \(R\) is a commutative ring (with unity) and let \(U\) be the set of all units in \(R\).
   (a) Prove that \(U\) is a multiplicative group.
   (b) Prove that \(U\) cannot contain zero divisors.
   (c) Describe \(U\) for the ring \(\mathbb{Z}_m\) and the polynomial ring \(\mathbb{R}[x]\).

5. Partition \(U_{17}\) into cosets of \((13)\).

6. Consider the set permutations on \(n\) elements \(\{1, 2, \ldots n\}\) (with \(n \geq 2\)) that keep the element 1 fixed: \(H = \{\sigma \in S_n: \sigma(1) = 1\}\). Prove that \(H\) is a subgroup of \(S_n\) and express the set of permutations that take 1 to 2: \(K = \{\sigma \in S_n: \sigma(1) = 2\}\) as a coset of \(H\).

7. Prove that among the residues modulo \(m\) it is exactly those that are coprime to \(m\) that are units in the ring \(\mathbb{Z}_m\).

8. Find the solution set for the system of congruences
\[
\begin{align*}
5x &\equiv 2 \pmod{48} \\
7x &\equiv 22 \pmod{30}
\end{align*}
\]

9. Exhibit (with proof) a surjective group homomorphism from the general linear group of invertible linear operators on the real plane under composition (or equivalently, \(2 \times 2\) nonsingular matrices with real coefficients under matrix multiplication) \(\text{GL}_2(\mathbb{R})\) to the multiplicative group of nonzero real numbers \(\mathbb{R}^*\). What is this homomorphism’s kernel?