1. (40 pts.)
   (a) Derive the Archimedean law from the Dedekind axiom for the real numbers, i.e. use the fact that any subset of \( \mathbb{R} \) which is bounded above has a supremum to show that for any \( a, b \in \mathbb{R}, a, b > 0 \exists n \in \mathbb{N} \) such that \( na > b \).
   (Hint: Consider the set of all \( na \))
   (b) Suppose \((X, d)\) is a metric space. Let \( D \subseteq X \). Prove that \( D \) is dense in \( X \), i.e. \( \overline{D} = X \iff (\forall \text{ open } U \subseteq X) \ D \cap U \neq \emptyset \).
   (c) Show that any compact metric space is separable, i.e. has a countable dense subset.
   (d) Show that \( \mathbb{Q} \) is dense in \( \mathbb{R} \), so \( \mathbb{R} \) is separable.
   (Hint: Use the Archimedean property and part (b))

2. (20 pts.) Let \( \mathcal{L}(E) \) denote the set of all limit points of a set \( E \) and \( \overline{E} \) denote the closure of \( E \). Show that
   (a) \( \mathcal{L}(\mathcal{L}(E)) \subseteq \mathcal{L}(E) \).
   (b) \( \mathcal{L}(E) = \mathcal{L}(\overline{E}) \).

3. (20 pts.) Suppose \( E \subseteq K \), where \( K \) is compact, and \( \mathcal{L}(E) = \emptyset \). Show that \( E \) is finite.

4. (20 pts.) Find all cluster points for the following sequences:
   (a) \( \left( 1 + \frac{2}{3n} \right)^{4n} \)
   (b) \( \left( \cos \left( \frac{n\pi}{4} \right) \right)^{(-1)^n} \)

5. (40 pts.) Determine whether the following series converge.
   (a) \( \sum_{k=1}^{\infty} \frac{3^k + 4^k}{5^k} \)
   (b) \( \sum_{k=1}^{\infty} \frac{k!}{k^k} \)
   (c) \( \sum_{k=1}^{\infty} \frac{(-2)^k k^2}{k!} \)
   (d) \( \sum_{k=1}^{\infty} \frac{\pi}{k} \)

6. (20 pts.) Suppose \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions. Show that the set \( \{x : f(x) = g(x)\} \) is closed in \( \mathbb{R} \).

7. (20 pts.) Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfies \( |f(x)| \leq |x| \) for all \( x \). What is \( f(0) \)? Prove that \( f \) is continuous at \( x = 0 \).

8. (20 pts.) Classify all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) which are continuous and such that \( f(\mathbb{R}) \in \mathbb{Q} \).