Throughout, suppose \((x_n)\) and \((y_n)\) are sequences of real numbers, \(x, y \in [-1, 1]\), \(x\) is a partial limit of \((x_n)\) and \(y\) is a partial limit of \((y_n)\).

1. (20 pts.) Determine whether each of the following statements is true in general. If true, prove it. If false, give a specific counterexample.

   (a) If \(\exists m \forall n \geq m \ x_n \geq y_n\), then \(x \geq y\).
   (b) If \(\forall m \exists n \geq m \ x_n \geq y_n\), then \(x \geq y\).
   (c) If 0 is a partial limit of \(x_n - y_n\), then \(x\) is a partial limit of \((y_n)\).
   (d) If \(x_n - y_n \to 0\), then \(x\) is a partial limit of \((y_n)\).

2. (10 pts.) Suppose \(A \subseteq \mathbb{R}\) and \(a \in \overline{A}\). Prove that there exists a sequence in \(A\) that converges to \(a\).

3. (10 pts.) Sketch the following functions and prove that they have no limit at 0:

   (a) \(f: (0, \infty) \to \mathbb{R}, f(x) = \sin \left(\frac{1}{x}\right)\)
   (b) \(f: \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}\)

4. (15 pts.) Let \(f\) be as in (3b) and let \(g(x) = f(x) \sin x\).

   (a) Sketch \(g\).
   (b) Prove that \(g\) is continuous at 0.
   (c) Find the set \(\{a \in \mathbb{R} : g\text{ is continuous at } a\}\).

5. (20 pts.) Suppose \(S \subseteq \mathbb{R}\) and \(f: S \to \mathbb{R}\) is a function. Determine whether each of the following statements is true in general. If true, prove it. If false, give a specific counterexample.

   (a) If \(S = \mathbb{R}\), \(f\) is continuous, and \(A\) is closed in \(\mathbb{R}\), then \(f^{-1}(A)\) is closed in \(\mathbb{R}\).
   (b) If \(S = \mathbb{R}\), \(f\) is continuous, and \(A\) is closed in \(\mathbb{R}\), then \(f(A)\) is closed in \(\mathbb{R}\).
   (c) If \(S = \mathbb{R}\), \(f\) is continuous, and \(\forall x \in \mathbb{Q} \ f(x) = 0\), then \(f \equiv 0\).
   (d) If \(S = (0, 1)\) and \(f\) is continuous, then \(\exists\) continuous \(g: [0, 1] \to \mathbb{R}\) such that \(\forall x \in (0, 1) \ g(x) = f(x)\).

6. (10 pts.) Prove that the equation \((x^2 + 1)^{-2} = x\) has a real solution.

7. (10 pts.) Suppose \(f: \mathbb{R} \to \mathbb{R}\) is decreasing and \(a \in \mathbb{R}\). Prove that \(\lim_{x \to a^-} f(x)\) exists.