1. (20 pts.) Prove that the following functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ have no limit in $[-\infty, \infty]$ at 0:
   
   (a) $f(x) = \frac{1}{x}$  
   (b) $f(x) = \cos\left(\frac{1}{x}\right)$

2. (20 pts.) Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Find the set $\{a \in \mathbb{R} : f \text{ is continuous at } a\}$. Prove your assertion.

3. (30 pts.) Suppose $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ is a function. Determine whether each of the following statements is true in general. If true, prove it. If false, give a specific counterexample.

   (a) If $S = \mathbb{R}$, $f$ is continuous, and $A$ is an open subset of $\mathbb{R}$, then $f^{-1}(A)$ is an open subset of $\mathbb{R}$.

   (b) If $S = \mathbb{R}$, $f$ is continuous and $A \subseteq \mathbb{R}$, then $\overline{f(A)} = f(\overline{A})$.

   (c) If $S = \mathbb{R}$, $f$ is continuous, and $A$ is a closed subset of $\mathbb{R}$, then $f(A)$ is a closed subset of $\mathbb{R}$.

   (d) If $S$ is finite, then $f$ is continuous.

   (e) If $S = (0, 1)$ and $f$ is continuous, then $f(S)$ has a minimum.