Name: ________________________________

Please show all work and justify your answers.

1. Suppose $G$ is a multiplicative group, $a \in G$. Prove that $a^n = e \iff |a|$ divides $n$.

2. Prove that any group with prime order must be cyclic.

3. How many distinct group automorphisms of $\mathbb{Z}$ are there? Explain. What about $\mathbb{Z}_p$?

   Hint: think about generators and their possible values under an automorphism.

4. Let $L$ be a line in $\mathbb{R}^3$ through the origin and let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection to $L$. Describe $\ker \varphi$ and its cosets in $\mathbb{R}^3$ geometrically. Sketch.

5. Suppose $H$ and $K$ are subgroups of a finite group $G$ and one of them is normal in $G$. Let $HK$ denote the set of all products $\{hk : h \in H, k \in K\}$. Prove that $HK < G$.

   Hint: consider a product $hkh'k$ and observe that $kh'$ belongs to $Kh'$ and $kH$.

6. Suppose $\varphi : R \to S$ is a homomorphism of rings. Prove that $\ker \varphi$ is an ideal of $R$. Show by example that $\varphi(R)$ is not necessarily an ideal of $S$. What hypothesis on $\varphi$ would ensure that $\varphi(R)$ is an ideal of $S$? Prove it.

7. Let $R = \mathbb{R}[x]$. Which elements of $R$ are units? Are there nonzero zero divisors in $R$? Let $A = \{p(x) \in R : p(0) = 0\}$. Prove that $A$ is an ideal of $R$. Is $A$ a prime ideal? Maximal? Prove your assertions.

8. Prove that any cubic polynomial in $\mathbb{R}[x]$ is reducible. Are there irreducible cubic polynomials in polynomial rings in one variable over other fields? Explain.

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