1. (20 pts.) Suppose $R$ and $S$ are rings and $f : R \to S$ is a ring homomorphism.
   (a) Show that $\ker f = \{ r \in R : f(r) = 0 \}$ is a two-sided ideal of $R$.
   (b) Show that if $f$ is onto and $I$ is a left ideal of $R$,
       then $f(I) = \{ s \in S : \exists r \in R \ f(r) = s \}$ is a left ideal of $S$.

2. (50 pts.) Suppose $R$ is a commutative ring with 1. For each of the following subsets of $R$ prove or disprove that it is closed under multiplication:
   (a) The set of units of $R$.
   (b) The set of nonunits of $R$.
   (c) The set of nonzero elements of $R$.
   (d) A prime ideal.
   (e) The complement of a prime ideal.

3. (60 pts.) Let $R$ be the ring of all continuous real valued functions of a real variable, i.e. $R = \{ f : \mathbb{R} \to \mathbb{R} : f$ is continuous $\}$, where addition and multiplication of functions are pointwise, i.e. $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$.
   Given a subset of the real line $V \subseteq \mathbb{R}$ define $I(V)$ to be the set of all continuous functions that vanish on $V$, i.e. $I(V) = \{ f \in R : \forall x \in V \ f(x) = 0 \}$.
   (a) Which functions are the units of $R$?
   (b) Prove or disprove: $R$ is an integral domain.
   (c) Show that if $V \subseteq \mathbb{R}$, then $I(V)$ is an ideal of $R$.
   (d) What are $I(\emptyset)$ and $I(\mathbb{R})$?
   (e) Show that if $a \in \mathbb{R}$, then $I(\{a\})$ is a prime ideal of $R$.
   (f) Show that if $a, b \in \mathbb{R}$ and $a \neq b$, then $I(\{a, b\})$ is not a prime ideal of $R$.

4. (40 pts.) True/false questions. Justification (proof or counterexample) required.
   T F (a) Every finite integral domain is a field.
   T F (b) If $R$ is an integral domain and $S = R \setminus \{0\}$, then $S^{-1}R$ is a field.
   T F (c) Every ideal of $\mathbb{Z}$ is a principal ideal.
   T F (d) Every ideal of the polynomial ring $\mathbb{Z}[x]$ is a principal ideal.