1. An exponentially growing yeast culture doubles in 7 days. How long would it take it to quadruple in size?

In the first 7 days you get double. In the next 7 days you get double that, which is 4 times the size. 

\[ \text{Quadrupling time is a fortnight.} \]

\[ b(t) = b_0 e^{kt} \text{ for some } k, \text{ where } b_0 \text{ is the initial size.} \]

Since doubling time is 7 days, \[ 2b_0 = b_0 e^{7k} \] 

\[ \ln 2 = 7k \quad \text{so} \quad k = \frac{\ln 2}{7} \]

To find quadrupling time set \[ 4b_0 = b_0 e^{kt} \]

and solve for \( t \): \[ \ln 4 = kt \]

\[ t = \frac{\ln 4}{k} = \frac{\ln 4}{\ln 2} = \frac{2 \ln 2}{\ln 2} = 2 \ln 2 \approx 1.4 \]

2. A population of bacteria grows exponentially according to \( b(t) = e^{2t} \). Find and illustrate on a graph:

(a) Population at \( t = 0 \) and \( t = 1 \).

(b) The average rate of change between \( t = 0 \) and \( t = 1 \).

(c) The instantaneous rates of change at \( t = 0 \) and \( t = 1 \).

3. Find the derivatives of

(a) \( \cos(1 + e^{2x}) \)  
(b) \( \ln(\ln x) \)

a) \[ [\cos(1 + e^{2x})]' = -\sin(1 + e^{2x}) \cdot e^{2x} \cdot 2 \]

b) \[ [\ln(\ln x)]' = \frac{1}{\ln t} \cdot \frac{1}{t} \]
4. Find the second derivative of the Hill function \( \frac{x^2}{1 + x^2} \) and use it to describe the curvature of the Hill function's graph.

\[
\begin{align*}
\text{If } h &= \frac{x^4}{1 + x^2}, \\
h' &= \frac{(x^2)'(1+x^2) - x^2(1+x^2)'}{(1+x^2)^2} = \frac{2x - 2x^3}{(1+x^2)^2}
\end{align*}
\]

\[
\begin{align*}
h'' &= 2 \frac{x'(1+x^2)^2 - x[(1+x^2)']}{(1+x^2)^4} \\
&= 2 \frac{(1+x^2)^2 - x(2+2x^2)}{(1+x^2)^4} \\
&= 2 \frac{1+x^2 - 4x^2}{(1+x^2)^3} = \frac{1-3x^2}{(1+x^2)^3}
\end{align*}
\]

For \( x > 0 \), \( h'' > 0 \) \( \iff \) \( 3x^2 < 1 \) \( \iff \) \( x^2 < \frac{1}{3} \) \( \iff \) \( x < \frac{1}{\sqrt{3}} \approx 0.577 \)

\[ \therefore \text{h curves up until about } x = 0.577 \text{ and then curves down} \]

5. The amount of medication \( M \) in the bloodstream of a patient on an intravenous drip is governed by the discrete dynamical system \( M_{t+1} = M_t - \frac{M_t}{2 + M_t} M_t + 1 \), where \( d \) is the rate of delivery through the drip and \( f(M_t) \) is the fraction of the medication absorbed by the patient. If \( f(M_t) = M_t/(2 + M_t) \) and \( d = 1 \), find the biologically significant equilibrium and determine its stability.

\[
M_{t+1} = M_t - \frac{M_t}{2 + M_t} M_t + 1
\]

To find the equilibrium set \( M = M - \frac{M^2}{2 + M} = 1 \),

\[
\frac{M^2}{2 + M} = 1
\]

\[
M^2 = 2 + M
\]

\[
M^2 - M - 2 = 0
\]

By inspection \( M = 2 \), \( M = -1 \) (not significant).

\[
\left[ M - \frac{M^2}{2 + M} + 1 \right]' = 1 - \frac{(M^2)'(2+M) - M^2(2+M)'}{(2+M)^2}
\]

\[ = 1 - \frac{2M(2+M) - M^2}{(2+M)^2} = 1 - \frac{4M + 2M^2 - M^2}{(2+M)^2} = 1 - \frac{4M + M^2}{(2+M)^2} \]

Eval \( @ M = 2 \) : \[ 1 - \frac{8 + 4}{4^2} = 1 - \frac{12}{16} = 1 - \frac{3}{4} = \frac{1}{4} < 1 \]

\[ \therefore M = 2 \text{ is a stable equilibrium} \]