1. Computing orders in \( U_7 \) we look for primitive elements (those of order \( \varphi(7) = 6 \))

\[
\text{? for(k=1,6,print("ord(["k, "]=" ,znorder(Mod(k,7)))))}
\]

\[
\text{ord(1)=1} \\
\text{ord(2)=3} \\
\text{ord(3)=6} \\
\text{ord(4)=3} \\
\text{ord(5)=6} \\
\text{ord(6)=2}
\]

Therefore, 3 and 5 are generators of \( U_7 \). For example,

\[
\text{? for(k=1,6,print("3^" ,k," = " ,Mod(3,7)^k))}
\]

\[
3^1=\text{Mod(3, 7)} \\
3^2=\text{Mod(2, 7)} \\
3^3=\text{Mod(6, 7)} \\
3^4=\text{Mod(4, 7)} \\
3^5=\text{Mod(5, 7)} \\
3^6=\text{Mod(1, 7)}
\]

Computing the orders of elements of \( U_8 = \{1, 3, 5, 7\} \) we find no elements of order 4 (other than 1 they have order 2), so \( U_8 \) is not cyclic.

2. In \( U_{13} \) we have \( 3^2 = 9, \; 3^3 = 1 \), so the subgroup generated by 3 is \( H = \{1, 3, 9\} \). By Lagrange’s theorem there are 4 distinct cosets \( H = \{1, 3, 9\}, \; 2H = \{2, 6, 5\}, \; 4H = \{4, 12, 10\}, \; 7H = \{7, 8, 11\} \).

3. \( \text{ker } f = \{0, 3, 6, 9, 12\} \) and the image is \( \{0, 5, 10\} \). Both are subgroups of \( \mathbb{Z}_{15} \), the kernel generated by 3 and the image by 5.

4. The second congruence implies the first, so we are reduced to solving \( x \equiv 5 \mod 8, \; x \equiv 3 \mod 5 \), where the two moduli are relatively prime. Following the book’s notation we have \( a_1 = 5, \; m_1 = 8, \; k_1 = 5, \; a_2 = 3, \; m_2 = 5, \; k_2 = 8 \). To obtain multiplicative inverses we need a Bezout relation. Euclid’s algorithm gives \( 8 = 5 + 3, \; 5 = 3 + 2, \; 3 = 2 + 1 \). Solving for remainders we get \( 3 = 8 - 5, \; 2 = 5 - 3, \; 1 = 3 - 2 \). Back substitution gives \( 1 = 3 - 2 = 3 - (5 - 3) = 2 \cdot 3 - 5 = 2 (8 - 5) - 5 = 2 \cdot 8 - 3 \cdot 5 \). Thus, the multiplicative inverses of \( k_i \) modulo \( m_i \) are \( r_1 = -3, \; r_2 = 2 \), so by the Chinese Remainder Theorem we have the unique (modulo \( m_1 m_2 \)) solution \( x = a_1 k_1 r_1 + a_2 k_2 r_2 = -5 \cdot 5 - 3 + 3 \cdot 8 - 2 = -27 \equiv 13 \mod 40 \).

Check: \( 13 \equiv 5 \mod 8 \) and \( 13 \equiv 3 \mod 5 \).

5. By long division \( x^3 - x = x(x^2 - 1) + x - 1 \). Since \( x - 1 \) divides \( x^2 - 1 \), the gcd is \( x - 1 \). To obtain a Bezout relation, solve for the remainder \( x - 1 = (x^3 - 1) - x(x^2 - 1) \).