1. Use the definition to compute the Laplace transform of $te^{-2t} u(t - 3)$.

For which $s$ does the transform converge?

$$
\int_0^\infty e^{-st} te^{-2t} u(t - 3) \, dt = \int_3^\infty e^{-(s+2)t} \, dt = \left[ \frac{te^{-(s+2)t}}{s + 2} \right]_3^\infty + \int_3^\infty \frac{e^{-(s+2)t}}{s + 2} \, dt
$$

$$
= \left[ \frac{te^{-(s+2)t}}{s + 2} - \frac{e^{-(s+2)t}}{(s + 2)^2} \right]_3^\infty = \left[ \frac{-e^{-(s+2)t}}{(s + 2)^2} [t(s + 2) - 1] \right]_{t=3}^{t=\infty} \rightarrow \frac{e^{-3s-6}(3s + 5)}{(s + 2)^2} \quad \Rightarrow s > -2
$$

2. Find the inverse Laplace transform of $\ln(s - 4)$.

Let $F = \ln(s - 4)$. Since $\mathcal{L}[t^n f] = (-1)^n \frac{d^n F}{ds^n}$, we have

$$
\mathcal{L}[tf] = -\frac{dF}{ds} = -\frac{1}{s - 4} = \mathcal{L}[-e^{4t}], \quad \text{so } tf = -e^{4t}, \quad \text{so } f = -\frac{e^{4t}}{t}
$$

3. Use the method of Laplace transforms to solve the initial value problem

$$
x'' + x = u(t - 3), \quad x(0) = 1, \quad x'(0) = 2
$$

Take $\mathcal{L}$: $s^2 X - s - 2 + X = -\frac{e^{-3s}}{s}$, so $(s^2 + 1)X = \frac{e^{-3s}}{s} + s + 2$. Solve for $X$:

$$
X = \frac{e^{-3s}}{s(s^2 + 1)} + \frac{s + 2}{s^2 + 1} = e^{-3s} \left[ \frac{1}{s} - \frac{s}{s^2 + 1} \right] + \frac{s}{s^2 + 1} + \frac{2}{s^2 + 1}
$$

Thus, $x = [1 - \cos(t - 3)] u(t - 3) + \cos(t) + 2 \sin(t)$

4. Find the Taylor series about $t = 0$ of $t^5 (4 + t^4)^{-1}$. Use the summation notation, but also write out the first three nonzero terms. What is the radius of convergence? Explain.

Let $x = -t^2/4$. Then $t^2 = -4x$, so

$$
\frac{t^5}{4 + t^2} = \frac{t^5}{4 - 4x} = \frac{1}{4} \cdot \frac{1}{1 - x} = \frac{t^5}{4} \sum_{k=0}^\infty x^k = \frac{t^5}{4} \sum_{k=0}^\infty \left( -\frac{t^2}{4} \right)^k = \sum_{k=0}^\infty \frac{(-1)^k}{4^{k+1}} t^{2k+5} = \frac{1}{4} t^5 - \frac{1}{16} t^7 + \frac{1}{64} t^9 + ...
$$

The nearest singularities to the origin are $2i$ and $-2i$, so the radius of convergence is 2.

Alternately you can use the ratio test:

$$
\frac{|t|^2}{4} < 1 \iff |t| < 2
$$

5. Find the first three nonzero terms of the power series solution about $t = 0$ to the initial value problem

$$(t + 1)x'' - x = 0, \quad x(0) = 0, \quad x'(0) = 2
$$

Let $x = \sum_{k=0}^\infty a_k t^k$, where $a_0 = 0$, $a_1 = 2$. Then $x'' = \sum_{k=0}^\infty a_{k+2}(k + 2)(k + 1)t^k$,

so $tx'' = \sum_{k=0}^\infty a_{k+2}(k + 2)(k + 1)t^k + \sum_{k=1}^\infty a_k(k + 1)kt^k = \sum_{k=0}^\infty a_{k+1}(k + 1)kt^k$

Plug into $tx'' + x'' - x = 0$, collect the coefficients of $x^k$ to obtain the recurrence relation $a_{k+1}(k + 1) + a_{k+2}(k + 2)(k + 1) - a_k = 0$, and solve:

$$
a_{k+2} = \frac{a_k - a_{k+1}(k + 1)k}{(k + 2)(k + 1)}, \quad \text{i.e. } a_k = \frac{a_{k-2} - a_{k-1}(k - 1)(k - 2)}{k(k - 1)}
$$

Choosing $k = 2, 3, ...$ we can obtain further coefficients: $a_2 = \frac{a_0 - a_1 \cdot 1 \cdot 0}{2 \cdot 1} = 0$, $a_3 = \frac{a_1 - a_2 \cdot 2 \cdot 1}{3 \cdot 2} = \frac{1}{3}$, $a_4 = \frac{a_2 - a_3 \cdot 3 \cdot 2}{4 \cdot 3} = -\frac{1}{6}$, so $x = 2t + \frac{1}{3} t^3 - \frac{1}{6} t^4$...