1. Suppose $G$ is a group such that every nontrivial element of $G$ has order 2. Prove that $G$
is abelian. Give an example of such a group that is not isomorphic to $\mathbb{Z}_2$.

$$\forall x \in G \quad x^2 = e, \quad \text{so} \quad x = x^{-1}.$$  

Let $a, b \in G$, then $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$  

Example: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

2. Prove that a group whose order is a prime must be cyclic.

Suppose $|G| = p$ (a prime). Since $p > 1$, $G$ has a nontrivial element $x$.

By Lagrange's theorem $|x| = |\langle x \rangle|$ divides $|G| = p$.

Since $x \neq e$, $|x| \neq 1$, so $|x| = p$.

$$\langle x \rangle = G$$

3. Let $H = \{ \alpha \in S_n : \alpha(1) = 1 \}$ with $n \geq 5$. Prove that $H$ is a subgroup of $S_n$. Prove or disprove that $H$ a normal subgroup of $S_5$.

Identity: $\varepsilon(1) = 1$  

Closure: If $\alpha, \beta \in H$, then $\alpha(1) = \beta(1) = 1$, so

$$\alpha \beta \varepsilon(1) = \alpha(\beta(1)) = \alpha(1) = 1.$$  

Inverse: If $\alpha \in H$, then $\alpha(1) = 1$, so $\alpha^{-1}(1) = 1$, so $\alpha^{-1} \in H$

$$H \text{ is a subgroup of } S_n$$

Extra credit: $H$ is not normal in $S_n$. 

Let $\alpha = (23)$, then $\alpha(1) = 1$, so $\alpha \notin H$

Let $\delta = (12)$. Then $\forall \delta^{-1}(1) = \delta((\delta^{-1}(1)) = \delta(1) = 1$, so $\delta^{-1} \notin H$.

$$\therefore$$
4. Let $H$ be as in the preceding problem. Suppose $\beta, \gamma \in S_n$ with $\beta(1) = \gamma(1)$. Prove that $\beta$ and $\gamma$ belong to the same left coset of $H$.

$$\beta = \gamma \gamma^{-1} \beta, \quad \text{but} \quad \gamma^{-1} \beta(1) = \gamma^{-1}(\gamma(1)) = \gamma^{-1}(\gamma(1)) = 1$$

So $\gamma^{-1} \beta \in H$. \[\Rightarrow \beta \in \gamma H \quad \square\]

5. Suppose $G$ is an abelian group whose order is odd. Prove that $\varphi : G \rightarrow G$ given by $\varphi(x) = x^2$ is an automorphism of $G$.

$$\varphi(xy) = (xy)^2 = xyxy = x^2y^2 = \varphi(x)\varphi(y)$$

$\uparrow$ abelian

$\therefore \varphi$ is a hom.

Suppose $\varphi(x) = x^2 = e$. By Lagrange's theorem,

$$|x| = \langle x \rangle \text{ divides } |G|, \quad \text{but } |G| \text{ is odd,}$$

so $|x| \neq 2$, so $|x| = 1$, so $x = e$.

$\therefore \varphi$ is $1-1$.

Since $G$ is finite, $\varphi$ is automatically onto. \[\square\]

$\therefore \varphi \in \text{Aut } G$. 