1. Find the isomorphism class of $U(12)$ as a finite abelian group.

$U(12) = \{ x \in \mathbb{Z}_{12} : \gcd(x, 12) = 1 \} = \{ 1, 5, 7, 11 \}$

By the classification theorem, $U(12) \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Since $5^2 \equiv 7^2 \equiv 11^2 \equiv 1 \mod 12$, $U(12)$ has no elements of order 4, so $U(12) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

2. Find all ideals of $\mathbb{Z}_{60}$. Explain why that's all of them. Draw a lattice (i.e. sketch subset relations among the ideals).

Since $\mathbb{Z}_{60}$ is a cyclic additive group, its ideals are cyclic subgroups and correspond to the divisors of 60

```plaintext
> numtheory[divisors](60);
{ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 }
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3. Prove that \( \{ \sigma \in S_3 \mid \sigma(3) = 3 \} \) is a subgroup of \( S_3 \). Is it abelian? Is it a normal subgroup of \( S_3 \)? Prove your assertions.

a) \( \sigma(1) \) takes 3 to 3, \( \sigma(1) \in H \)

b) \( \sigma, \tau \in H \), \( \sigma(3) = 3, \tau(3) = 3 \), so \( \sigma \tau \in H \)

c) \( \sigma \in H \), \( \sigma(3) = 3 \), so \( \sigma^{-1}(3) = 3 \), so \( \sigma^{-1} \in H \).

\[ H = \{ (1), (12) \} \text{ is abelian (} H \cong \mathbb{Z}_2 \) \]

Let \( \tau = (23) \). Then \( \tau^{-1} = (23) \)

\[ (23)(12)(23) = (13) \notin H, \text{ so } \tau^{-1}H \tau \not\subseteq H, \text{ so } H \not\triangleleft S_3 \]

4. Find the quotient and remainder of \( x^4 + 3x^3 + 2x^2 + x - 1 \) divided by \( 2x^2 + 1 \) in \( \mathbb{Z}_7[x] \).

\[
> p := x^4 + 3x^3 + 2x^2 + x - 1; \\
> q := 2x^2 + 1; \\
> \text{quo}(p, q, x, 'r') \mod 7; r \mod 7; \\
> 4x^2 + 5x + 6 \\
> 3x
\]

5. Let \( A = \{ p \in \mathbb{R}[x] \mid p(0) = 0 \} \). Prove that \( A \) is an ideal of \( \mathbb{R}[x] \). Is \( A \) a prime ideal? Maximal? Explain.

Define \( \varphi : \mathbb{R}[x] \to \mathbb{R} \) by \( \varphi(p(x)) = p(0) \)

By inspection, \( \varphi \) is a ring homomorphism with kernel \( \varphi = A \) and \( \varphi(\mathbb{R}[x]) = \mathbb{R} \)

By the 1st isomorphism theorem \( \frac{\mathbb{R}[x]}{A} \cong \mathbb{R} \)

Since \( \mathbb{R} \) is a field, \( A \) is a maximal ideal (thus a prime ideal) \( \cap \mathbb{R}[x] \).