Approximation of elliptic boundary value problems

History:

- 1950’s: Finite differences and Rayleigh-Ritz-Galerkin
- FD: Young (1950) — over relaxation; faster iterative methods for large systems; 5-point schemes
- Courant: Variational method with piecewise linear basis functions leads to a 5-point scheme for the Laplace equation. (forgotten for 20 years)
- Decisive step: engineers independently develop finite elements (piecewise polynomial shape functions leads to FD)

Requirements for an approximation:

- stability and optimal stability of approximate problems
- convergence of solutions, uniformity and optimal speed of convergence
- minimization of error
- sparsity and optimal condition number of matrices

Domain: $\Omega \subseteq \mathbb{R}^n$ — bounded open subset with smooth boundary $\Gamma$.

Differentiation: Let $p \in \mathbb{Z}_+^n$ with 1-norm. Define $D^p := \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}$. 

Space of test functions: $\mathcal{D}(\Omega) = \{u \in C^\infty(\Omega) \text{ with compact support in } \Omega\}$.

For a distribution $f$ define $\frac{\partial f}{\partial x_i}$ by $\langle \frac{\partial f}{\partial x_i}, \varphi \rangle := \langle f, -\frac{\partial \varphi}{\partial x_i} \rangle \forall \varphi \in \mathcal{D}(\Omega)$.

Differential operator: $Au := \sum_{|p|,|q| \leq k} (-1)^{|q|} D^q[a_{pq}(x)D^pu], \text{ where } a_{pq} \in L^\infty(\Omega)$.

Normal boundary derivatives $\gamma_j u \ (\gamma_j \text{ is just restriction to } \Gamma)$.

Sobolev space: $H^s(\mathbb{R}^n) := \{ u \in L^2(\mathbb{R}^n) : (1 + |\eta|^2)^{\frac{s}{2}} \hat{u}(\eta) \in L^2(\mathbb{R}^n) \} = \{ u \in L^2(\mathbb{R}^n) : D^p u \in L^2(\mathbb{R}^n), |p| \leq s \}$

Let $H^s(\Omega)$ be the space of restrictions to $\Omega$ of functions in $H^s(\mathbb{R}^n)$. $H^s_0(\Omega) := \text{closure of } \mathcal{D}(\Omega) \text{ in } H^s(\Omega)$. $H^s(\Gamma) \cong H^s(\mathbb{R}^{n-1})$.

Trace theorem: $\gamma := (\gamma_0, \ldots, \gamma_{s-1}), H^s(\Omega) \rightarrow \prod_{j=0}^{s-1} H^{s-j-\frac{1}{2}}(\Gamma)$ is a bounded linear operator and $\ker \gamma = H^s_0(\Omega)$.

Energy product: a bilinear form $a(u, v) := \sum_{|p|,|q| \leq k} \int_{\Omega} a_{pq}(x) D^p u D^q v \, dx$

$H^k(\Omega, \Lambda) := \{ u \in H^k(\Omega) : \Lambda u \in L^2(\Omega) \}$

Green’s formula: $\exists$ linear operators $\delta_j : H^{k}(\Omega) \rightarrow H^{k-j-\frac{1}{2}}(\Gamma)$ $(k \leq j \leq 2k-1)$ such that $\forall u \in H^k(\Omega, \Lambda), v \in H^k(\Omega)$

\[ a(u, v) = \int_{\Omega} \Lambda u \cdot v \, dx + \sum_{j=0}^{k-1} \int_{\Gamma} \delta_{2k-j-1} u \gamma_j v \, d\sigma(x) \]

Neumann problem: Given a forcing function $f \in L^2(\Omega)$ and boundary conditions $t_j \in H^{k-j-\frac{1}{2}}(\Gamma), k \leq j \leq 2k-1$, we look for $u \in H^{k}(\Omega, \Lambda)$ with $\Lambda u + \lambda u = f$ and $\delta_j u = t_j$.

Equivalent formulation: Let $\langle u, v \rangle := \int_{\Omega} u(x) v(x) \, dx, \langle f, g \rangle := \int_{\Gamma} f(x) g(x) \, d\sigma(x), \ell(v) := \langle f, v \rangle + \sum_{j=0}^{k-1} \langle t_{2k-j-1}, \gamma_j v \rangle$.

$u$ is a solution of the Neumann problem $\iff u \in H^k(\Omega)$ and $a(u, v) + \lambda(u, v) = \ell(v) \ \forall v \in H^k(\Omega)$

General problem: Suppose $V \subseteq H$ are Hilbert spaces, $V$ is compact and dense in $H$. Let $a$ and $\ell$ be continuous bilinear and linear forms on $V$. Find $u \in V$ such that $a(u, v) + \lambda(u, v) = \ell(v) \ \forall v \in V$.

Existence-uniqueness theorem: Suppose $a$ is $V$-elliptic, i.e. $a(v, v) \geq c|v|^2_V \ \forall v \in V$ and some constant $c$. If $\lambda$ is not in the spectrum of $a$ (a countable set of isolated points), then the solution exists and is unique.

Proof: The result follows from the Lax-Milgram theorem and the Riesz-Fredholm theorem.

References:


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