Multinomial Formula by Dr. Dmitry Gokhman 1995

\[(a_1 + \ldots + a_m)^n = \sum_{i_1 + \ldots + i_m = n} C(n, i_1, \ldots, i_m) a_1^{i_1} \ldots a_m^{i_m},\]

where we assume \(i_k \geq 0\) and the multinomial coefficients are

\[C(n, i_1, \ldots, i_m) = \frac{n!}{i_1! \ldots i_m!}.\]

Using multi-index notation (e.g. \(I = (i_1, \ldots, i_m), |I| = \sum_{k=1}^{m} i_k\)) the multinomial formula can written more compactly

\[\left(\sum_{k=1}^{m} a_k\right)^n = \sum_{|I|=n} C(n, I) \prod_{k=1}^{m} a_k^{i_k}, \quad C(n, I) = \frac{n!}{\prod_{k=1}^{m} i_k!}.\]

Example

The binomial formula is a special case

\[(a + b)^n = \sum_{i+j=n} C(n, i, j) a^i b^j, \quad C(n, i, j) = \frac{n!}{i! j!},\]

and one can obtain a more familiar form of this by substituting \(j = n - i\) and omitting \(j\) from the coefficient

\[(a + b)^n = \sum_{i=0}^{n} C(n, i) a^i b^{n-i}, \quad C(n, i) = \frac{n!}{i! (n-i)!}.\]

Recursion for the coefficients

Binomial coefficients are often calculated by a “Pascal triangle” recursion:

\[C(n, i, j) = C(n-1, i-1, j) + C(n-1, i, j-1)\]

or in more conventional notation with \(j\) dropped

\[C(n, i) = C(n-1, i-1) + C(n-1, i).\]

For multinomial coefficients there is a recursion corresponding to a higher dimensional object (triangle \(\rightarrow\) tetrahedron \(\rightarrow\) etc.):

\[C(n, i_1, \ldots, i_m) = C(n-1, i_1-1, i_2, \ldots, i_m) + C(n-1, i_1, i_2-1, \ldots, i_m) + \ldots + C(n-1, i_1, i_2, \ldots, i_m-1)\]