Energy, entropy and uniqueness

Wave equation in one spatial dimension: \( u_{tt} = c^2 u_{xx} \)

Boundary conditions: \( u(0, t) = u(L, t) = 0 \)

Initial conditions: \( u(x, 0) = f(x), \ u_t(x, 0) = g(x) \)

Solution by separation of variables and Fourier series: \( u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi}{L} x \right) \left[ B_n \cos \left( \frac{cn \pi}{L} t \right) + D_n \sin \left( \frac{cn \pi}{L} t \right) \right] \)

where \( B_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n \pi}{L} x \right) \, dx \) and \( D_n = \frac{2}{cn \pi} \int_0^L g(x) \sin \left( \frac{n \pi}{L} x \right) \, dx \)

Energy density: \( \varepsilon(x, t) = \frac{1}{2} (c^2 u^2_x + u^2) \)

You can think of the first term as kinetic energy density and the second as potential energy density.

Total energy: \( E(t) = \int_0^L \varepsilon(x, t) \, dx \)

Heat equation in three spatial dimensions: \( u_{tt} = c^2 \nabla^2 u \)

By Duhamel’s principle, the total heat in a small volume \( \Omega \) is \( Q \approx M u \, \text{vol}(\Omega) \), where \( M \) is specific heat of matter.

By Newton’s law of cooling, heat flux across the boundary \( \partial \Omega \) is proportional to temperature gradient: \( Q_t = N \int_{\partial\Omega} \nabla u \cdot \hat{n} \, dS \)

By the Gauss-Ostrogradski divergence theorem \( Q_t = N \int_{\Omega} \nabla \cdot \nabla u \, dV \approx N \nabla^2 u \, \text{vol}(\Omega) \)

Dividing \( M u \, \text{vol}(\Omega) \approx N \nabla^2 u \, \text{vol}(\Omega) \) by the volume and taking limit as \( \text{vol}(\Omega) \to 0 \) we obtain \( u_t = \frac{N}{M} \nabla^2 u \)

Entropy: Define entropy density \( \varepsilon = \frac{1}{2} u^2 \) and integrate over \( \Omega \): \( E(t) = \int_\Omega \varepsilon \, dV = \frac{1}{2} \int_\Omega u^2 \, dV \)

Entropy principle: In the presence of temperature gradients, total entropy of an insulated body decreases.

Product rule for divergence: \( \nabla \cdot (\varphi \Phi) = \nabla \varphi \cdot \Phi + \varphi (\nabla \cdot \Phi) \) implies \( \nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u (\nabla^2 u) \).

Thus, \( \varepsilon_t = u_t u + c^2 \nabla^2 u = c^2 [\nabla \cdot (u \nabla u) - (\nabla u) \cdot (\nabla u)] \).

Integrating and applying the divergence theorem we obtain \( E_t = c^2 \left[ \int_{\partial\Omega} u \nabla u \cdot \hat{n} \, dS - \int_{\Omega} (\nabla u) \cdot (\nabla u) \, dV \right] \).

For an insulated body \( \nabla u \cdot \hat{n} = 0 \) on the boundary \( \partial \Omega \), so \( E_t = -c^2 \int_{\Omega} (\nabla u) \cdot (\nabla u) \, dV \leq 0 \).

Uniqueness: Given two solutions with the same initial state, their difference \( u \) is a solution with initial state 0. Its initial entropy is 0. Since \( E \geq 0 \) and cannot increase (\( E_t \leq 0 \)), it stays 0.

Therefore, at any time, \( \nabla u = 0 \), so \( u \) is a constant and thus \( u = 0 \).