AN ASYMPTOTIC EXISTENCE THEOREM IN $\mathbb{C}$
FOR THE RICCATI EQUATION

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Abstract

We prove a generalization to $\mathbb{C}$ of a well known theorem [3] that in $\mathbb{R}$, the Riccati equation $W' + W^2 = F^2$ with $F$ real and positive on the positive real axis, such that $\lim_{x \to +\infty} F = +\infty$, has a family of solutions asymptotic to $F$ and a unique solution asymptotic to $-F$.

The Riccati equation $W' + W^2 = F^2$ in the complex domain, where $F$ is a holomorphic function in a partial neighborhood of infinity $D$ and $F(z) \to \infty$ as $z \to \infty$ in $D$, has uniformly approximate solutions $F$ and $-F$. Furthermore, we show that there exist solutions $W$ which are uniformly asymptotic to $\pm F$. We establish criteria for the shape of partial neighborhoods of infinity where this occurs.

There is a family of solutions uniformly asymptotic to $F$ and a unique solution uniformly asymptotic to $-F$ or vice versa. The situation is reversed in adjacent neighborhoods. We establish a criterion for determining the particular case. Specific examples are provided for the following cases:

1. $F$ is of polynomial or iterated logarithmic growth and the region is a sector (this is the classical case),
2. $F = e^x$ and the region is a horizontal strip,
3. $F = e^{e^x}$ and the region is funnel shaped.

1 INTRODUCTION

The underlying notion of asymptotic analysis of a differential equation near a singularity is that of a limiting process. The classical approach in $\mathbb{C}$ has been to consider uniform convergence in sectors or, more generally, convergence with respect to sectorial neighborhood
An asymptotic existence theorem in $C^2$ systems. In most cases considered so far, the growth of the coefficients in the differential equation has been at most polynomial. In this paper we shall analyze the Riccati equation

$$W' + W^2 = F^2, \quad \text{where } F \to \infty,$$

and the rate of growth of $F$ is arbitrary. For this purpose, it does not suffice to use sectors and we shall consider more general regions. We will however make a practical assumption that the partial neighborhood of the singularity (in our case infinity) is endowed with curvilinear coordinates $(p, q)$ satisfying certain properties.

Our limit process will be uniform convergence as one of the coordinates, $p$, tends to infinity and uniformity is with respect to the other coordinate, $q$. As is typical when dealing with uniform convergence, we shall restrict an open $^1$ domain

$$D = \{(p, q): p_0 \leq p < \infty, q_0 < q < q_1\},$$

to a proper closed subdomain of the form

$$\overline{D} = \{(p, q): p_0 \leq p < \infty, q_2 \leq q \leq q_3\}, \quad \text{where } q_2, q_3 \in (q_0, q_1).$$

In several examples we will compute explicitly the curvilinear coordinates $(p, q)$ and the corresponding expressions for the Riemann metric.

Asymptotic properties depend only on behavior near the point of interest (usually infinity). Following the notation in [3], the word ultimately will be used to describe properties of functions that hold for all sufficiently large values of the parameter (usually $x$ or $p$). For example, a function $f(x)$ is ultimately zero-free, when there exists an $x_0$ such that for all $x \geq x_0$ we have $f(x) \neq 0$.

The first step in asymptotic analysis of a differential equation is a search for approximate solutions. A function $G$ is considered to be an approximate solution, when after it is substituted into the equation, the resulting expression (the difference of the sides) tends to zero. It is desirable that approximate solutions be in some sense easy to calculate, at least no worse than the coefficients. For the Riccati equation (1) the situation is particularly simple as the approximate solutions are $F$ and $-F$.

Given an approximate solution $G$ to a differential equation one seeks exact solutions approximated by it with a substitution of $W = G(1 + \alpha)$ into the equation. This gives a first order equation for $\alpha$. For the Riccati equation (1), $F$ itself is an approximate solution ($-F$ is one as well) and the equation for $\alpha$ is

$$\alpha' + \left(2F + \frac{F'}{F}\right)\alpha + F\alpha^2 + \frac{F'}{F} = 0.$$  \hspace{1cm} (2)

$^1$Note that the terms open and closed refer not exactly to whether the domain is an open or closed set, but rather to the range of $q$. The actual point of interest $\infty$ is a limit point, but is not included in the domain. The topology at the outer edge is of little interest.
To solve this equation for $\alpha$ we shall use a method that has been used in classical theory for the same purpose (e.g. [4]), namely Newton’s method: an iterative procedure, where at each step the nonlinear terms are first thrown to the right hand side of the equation. The right hand side is then treated as known by using the answer from the preceding step and the resulting linear inhomogeneous equation is integrated using a special case of the variation of parameters formula, which we state and prove here for convenience:

**Theorem.** Given a first order linear inhomogeneous equation $Y' + fY = g$, the general solution is

$$Y = e^{-\int f} \left( C + \int g e^{\int f} \right) \tag{3}$$

**Proof:** If we fix an antiderivative of $f$ and try $Y = U e^{-\int f}$ so that $U = Y e^{\int f}$, then $U' = (Y' + fY) e^{\int f} = g e^{\int f}$, so $U = \int g e^{\int f} + C$. Note that $Y = C e^{-\int f}$ is the general solution to the homogeneous equation $Y' + fY = 0$. □

The iterations converge in a partial neighborhood of infinity. In the classical theory the region is a sector. In general, the shape of the neighborhood depends on the equation (in our case the rate of growth of $F$) and on the approximate solutions used. In fact, the entire punctured neighborhood is subdivided into such partial neighborhoods separated by lines (known as Stokes lines) which are given by roots of a certain indicial equation.

There is an arbitrary constant $C$ in the variation of parameters formula (3), which is kept fixed for a given iteration procedure. For different values of $C$ we obtain different solutions to the equation. Depending on the particular partial neighborhood and on whether we choose $G = F$ or $G = -F$, two qualitatively different situations occur. In one case (A) any value of $C$ leads to a solution $\alpha \to 0$. We thus obtain a family of solutions $W$ (parametrized by $C$) asymptotic to $G$. In the other case (B) only one value of $C$ leads to a solution $\alpha \to 0$, and we have only one solution $W \sim G$. In an adjacent neighborhood the correspondence between the choice of $G$ and the above two cases is reversed.

To determine whether case A or B holds in a given region we let $f = 2F + F'/F$ (so that $f \alpha$ is the linear term in equation (2)) and examine the asymptotic behavior of $e^{\int f}$, which largely depends on the sign of $\text{Re} \int f$. We express this criterion in terms of $\text{Re} \int F$.

Case A occurs when $\text{Re} \int F \to -\infty$ and case B when $\text{Re} \int F$ is bounded away from $-\infty$. Observe that the cases are reversed if $F$ is replaced by $-F$.

## 2 Curvilinear coordinates and l’Hôpital’s rule

This section is devoted to establishing certain elementary results needed for further exposition. Some elementary results are needed to apply l’Hôpital’s rule to quotients of real valued functions on domains in $\mathbb{C}$, which are real or imaginary parts or norms of complex integrals. The domains are equipped with curvilinear coordinates $(p, q)$ and we take limits
as $p \to \infty$. The derivative in l'Hôpital’s rule is with respect to $p$. To show that the limits are uniform with respect to $q$ we need to show that the application of l'Hôpital’s rule preserves uniformity. The quotients mentioned above are of exponential integrals and come from the variation of parameters formula. Taking the norm of an exponential integral gives the real part of the exponent, so we have to deal with real parts of integrals.

Suppose $(p, q), \ p \in [0, \infty), q \in [0, 1]$ is a system of coordinates on a region in $\mathbb{C}$.

**Lemma 2.1** Suppose $W(z) = \gamma(z)e^{i\varphi(z)}$ is a holomorphic function of $z = x + iy$. Then

$$\left|\frac{dW}{dz}\right| = (g_{11})^{-\frac{1}{2}} \sqrt{\left(\frac{\partial \gamma}{\partial p}\right)^2 + \gamma^2 \left(\frac{\partial \varphi}{\partial p}\right)^2},$$

where $g_{11} = \left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2$

**Proof:** The chain rule implies

$$\frac{\partial W}{\partial p} = W'(z) \frac{\partial z}{\partial p},$$

so

$$W'(z) = \frac{1}{\frac{\partial x}{\partial p}} \frac{\partial W}{\partial p} = \frac{1}{\frac{\partial x}{\partial p} + i \frac{\partial y}{\partial p}} \left(\frac{\partial \gamma}{\partial p} e^{i\varphi} + i \gamma e^{i\varphi} \frac{\partial \varphi}{\partial p}\right) = \frac{e^{i\varphi}}{\frac{\partial x}{\partial p} + i \frac{\partial y}{\partial p}} \left(\frac{\partial \gamma}{\partial p} + i \gamma \frac{\partial \varphi}{\partial p}\right).$$

Thus

$$|W'(z)| = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2}} \sqrt{\left(\frac{\partial \gamma}{\partial p}\right)^2 + \gamma^2 \left(\frac{\partial \varphi}{\partial p}\right)^2}.$$


There is another way to compute $g_{11}$, particularly suited to the situation when the coordinate system $(p, q)$ is orthogonal and expressed in terms of $(x, y)$. The quantity $g_{11}$ is recognizable as a component of the canonical flat metric on the tangent space of the Euclidean plane expressed in curvilinear coordinates $p, q$. For a discussion of the differential geometric point of view and of Riemannian metrics see, for example, Ch. 3 of [5] or Sec. 1.2 in [1].

In general, the corresponding change of basis in tangent space is

$$\frac{\partial}{\partial p} = \frac{\partial x}{\partial p} \frac{\partial}{\partial x} + \frac{\partial y}{\partial p} \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial q} = \frac{\partial x}{\partial q} \frac{\partial}{\partial x} + \frac{\partial y}{\partial q} \frac{\partial}{\partial y}.$$

Taking inner products gives a representation of the canonical flat metric as the matrix

$$(g_{ij}) = \left(\begin{array}{cc}
\left< \frac{\partial}{\partial p}, \frac{\partial}{\partial p} \right> & \left< \frac{\partial}{\partial p}, \frac{\partial}{\partial q} \right> \\
\left< \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \right> & \left< \frac{\partial}{\partial q}, \frac{\partial}{\partial q} \right>
\end{array}\right) = \left(\begin{array}{cc}
\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 & \frac{\partial x}{\partial p} \frac{\partial y}{\partial q} + \frac{\partial y}{\partial p} \frac{\partial x}{\partial q}
\\
\frac{\partial x}{\partial q} \frac{\partial y}{\partial p} + \frac{\partial y}{\partial q} \frac{\partial x}{\partial q} & \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2
\end{array}\right)$$
For examples of such representations in polar and spherical coordinates see Exp. 1 in Sec. 7 and Exc. 14.2 in [5].

The inverse matrix $g^{ij}$ represents the canonical flat metric on the cotangent space (see Exc. 5.11 in [5]). In the orthogonal case, the off-diagonal entries $g_{12} = g_{21} = 0$ are zero and the inverse is

$$
(g^{ij}) = \begin{pmatrix}
\langle dp, dp \rangle & \langle dp, dq \rangle \\
\langle dq, dp \rangle & \langle dq, dq \rangle
\end{pmatrix} = \begin{pmatrix}
\frac{1}{g_{11}} & 0 \\
0 & \frac{1}{g_{22}}
\end{pmatrix},
$$

so $g_{11} = 1/(dp, dp)$.

**Lemma 2.2** Suppose $W(z) = \gamma(z)e^{i\varphi(z)}$ is a holomorphic function of $z$ and $\Omega(z)$ is an antiderivative of $W$. Then

$$
\frac{\partial \text{Re } \Omega}{\partial p} = \sqrt{g_{11}} \gamma \cos(\varphi + \psi),
$$

where $\psi$ is the angle of incline of $p$, i.e. $\tan \psi = \left(\frac{\partial x}{\partial p}\right)^{-1} \frac{\partial y}{\partial p}$.

**Proof:** Since $W$ and its antiderivative $\Omega$ are holomorphic functions,

$$
W = X(z) + iY(z) = \frac{d}{dz} \Omega = \frac{\partial}{\partial x} \Omega = -i \frac{\partial}{\partial y} \Omega =
$$

$$
= \frac{\partial \text{Re } \Omega}{\partial x} + i \frac{\partial \text{Im } \Omega}{\partial x} = -i \frac{\partial \text{Re } \Omega}{\partial y} + \frac{\partial \text{Im } \Omega}{\partial y}.
$$

Comparing real and imaginary parts in the above equation shows that

$$
\frac{\partial \text{Re } \Omega}{\partial x} = X, \quad \frac{\partial \text{Re } \Omega}{\partial y} = -Y, \quad \text{so} \quad \frac{\partial \text{Re } \Omega}{\partial p} = X \frac{\partial x}{\partial p} - Y \frac{\partial y}{\partial p}.
$$

Since in polar notation

$$
\frac{\partial z}{\partial p} = \frac{\partial x}{\partial p} + i \frac{\partial y}{\partial p} = \left| \frac{\partial x}{\partial p} + i \frac{\partial y}{\partial p} \right| e^{i\psi} = \sqrt{g_{11}} (\cos \psi + i \sin \psi),
$$

we have

$$
\frac{\partial \text{Re } \Omega}{\partial p} = X \frac{\partial x}{\partial p} - Y \frac{\partial y}{\partial p} = \sqrt{g_{11}} (X \cos \psi - Y \sin \psi) =
$$

$$
= \sqrt{g_{11}} (\gamma \cos \varphi \cos \psi - \gamma \sin \varphi \sin \psi).
$$

Suppose $f, g$ are differentiable real valued functions in two variables. The following shows the uniformity of applying l’Hôpital’s rule (both cases) to obtain $f/g \to 0$.

First the case $0/0$. 

Lemma 2.3 Suppose $f, g : [x_1, x_0] \times [y_1, y_2] \to \mathbb{R}$ are differentiable functions such that $f(x_0) = g(x_0) = 0$ and $\frac{\partial}{\partial x} f, \frac{\partial}{\partial x} g$ are not simultaneously zero near $x_0$. Assume that for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $(x, y)$ satisfying $|x - x_0| < \delta$ we have

$$\left| \frac{\partial}{\partial x} f(x, y) \right| < \epsilon.$$

Then for any $\epsilon_1 > 0$, there is a $\delta_1 > 0$ such that for all $(x, y)$ satisfying $|x - x_0| < \delta_1$ we have

$$\left| \frac{f(x, y)}{g(x, y)} \right| < \epsilon_1.$$

Proof: By Cauchy’s Mean Value Theorem (see e.g. [2]), for each $x \in [x_1, x_0]$ there exists $\xi = \xi(x, y) \in [x, x_0]$ such that

$$\frac{f(x, y) - f(x_0, y)}{g(x, y) - g(x_0, y)} = \frac{\partial}{\partial x} f(\xi, y) \cdot g(\xi, y) + \frac{\partial}{\partial x} g(\xi, y) \cdot f(\xi, y).$$

Now let $\epsilon, \delta > 0$ such that

$$\left| \frac{\partial}{\partial x} f(x, y) \right| < \epsilon \quad \text{for} \quad (x, y) \in [x_0 - \delta, x_0] \times [y_1, y_2].$$

If $x \in [x_0 - \delta, x_0]$, then $\xi \in [x_0 - \delta, x_0]$. □

By the usual change of variables $x \mapsto 1/x$, the proof carries over to the situation where $x_0 = \infty$.

Now for the case ($\infty/\infty$), this time at $x_0 = \infty$. First, we need a definition and a lemma.

**Definition 2.1** A property of a function $f(x)$ holds ultimately, whenever it holds for all sufficiently large $x$, i.e. there exists $x_0$ such that the property holds for all $x \geq x_0$.

**Lemma 2.4** Suppose $f : [x_1, \infty) \times [y_1, y_2] \to \mathbb{R}$ is a differentiable function such that uniformly $\lim_{x \to \infty} f = \infty$ and $\frac{\partial}{\partial x} f \neq 0$ in $[\frac{x}{\delta}, \infty)$ for sufficiently small $\delta > 0$. Then for any fixed sufficiently large $\bar{x} \in [x_1, \infty]$

$$\frac{f(\bar{x}, y)}{f(x, y)} \to 0$$

uniformly.

Proof: Observe that

$$\frac{\partial}{\partial x} \left( \frac{f(\bar{x}, y)}{f(x, y)} \right) = -\frac{f(\bar{x}, y)}{f(x, y)^2} \frac{\partial f}{\partial x}(x, y).$$
Since ultimately \( f > 0 \) and \( \frac{\partial}{\partial y} f \neq 0 \), we have ultimately \( \frac{\partial}{\partial x} \left( \frac{f(\bar{x}, y)}{f(x, y)} \right) \neq 0 \). Thus, we can apply the Implicit Function Theorem to obtain a family of continuous functions \( \xi(x) \) satisfying \( f(\bar{x}, y)/f(x, y) = \xi(x) \).

Each function \( \xi(x) \) being continuous on a compact set \([y_1, y_2]\) possesses a maximum. By Darboux’s Intermediate Value Theorem for derivatives (see e.g. [2]), \( f(\bar{x}, y)/f(x, y) \) is ultimately a monotonic function of \( x \) in \([1/\delta, \infty)\) for some \( \delta > 0 \). Therefore, for any \( \epsilon > 0 \), \( |f(\bar{x}, y)/f(x, y)| < \epsilon \) on \([1/\delta_1, \infty) \times [y_1, y_2] \), where \( \delta_1 = \min\{\delta, 1/\max_{[y_1, y_2]} \xi(x)\} \). Thus, the convergence is uniform.

**Proposition 2.1** Suppose \( f, g: [x_1, \infty] \times [y_1, y_2] \to \mathbb{R} \) are differentiable functions such that \( \lim_{x \to \infty} f = \lim_{x \to \infty} g = 0 \) or \( \lim_{x \to \infty} f = \lim_{x \to \infty} g = \infty \) and for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for all \((x, y)\) satisfying \( |1/x| < \delta \) we have
\[
\left| \frac{\frac{\partial}{\partial x} f(x, y)}{\frac{\partial}{\partial x} g(x, y)} \right| < \epsilon.
\]

Then for any \( \epsilon_1 > 0 \), there is a \( \delta_1 > 0 \) such that for all \((x, y)\) satisfying \( |1/x| < \delta_1 \) we have
\[
\left| \frac{f(x, y)}{g(x, y)} \right| < \epsilon_1.
\]

**Proof:** The 0/0 case has already been covered, so we assume
\[
\lim_{x \to \infty} f = \lim_{x \to \infty} g = \infty.
\]

Let \( x_0 \in [x_1, \infty) \). Again by Cauchy’s Mean Value Theorem, for each \( x \in [x_1, x_0] \) there exists \( \xi = \xi(x, y) \in [x, x_0] \) such that
\[
\frac{\frac{\partial}{\partial x} f(\xi, y)}{\frac{\partial}{\partial x} g(\xi, y)} = \frac{f(x, y) - f(x_0, y)}{g(x, y) - g(x_0, y)} = \frac{f(x, y)}{g(x, y)} \left( \frac{1 - f(x_0, y) f(x, y)}{1 - g(x_0, y) g(x, y)} \right).
\]

It remains to show that the fraction in parentheses converges uniformly to 1. This follows easily, since as we have seen before, both \( f(x_0, y)/f(x, y) \) and \( g(x_0, y)/g(x, y) \) converge uniformly to 0. 

**3 Approximate solutions**

Dividing the Riccati equation \( W' + W^2 = F^2 \) by \( W^2 \) we obtain
\[
\frac{W'}{W^2} + 1 = \frac{F^2}{W^2}.
\]
For the real variable case we know that if \( W'/W^2 \rightarrow 0 \), then \( F^2/W^2 \rightarrow 1 \), so \( W \sim F \) or \( W \sim -F \). This is precisely the case if a solution of \( W' + W^2 = F^2 \) represents a germ \( W \) at \( \infty \) in a Hardy field. The following proposition gives an analogous result for the complex case.

**Proposition 3.1** Suppose \((p,q)\) is an orthogonal coordinate system such that for sufficiently large \( p_0 \) we have \( \sqrt{g_{11}} \) bounded away from 0 on \( \{(p,q): p > p_0\} \subseteq D \). Suppose \( F(z) = \gamma(z)e^{i\varphi(z)} \) is a holomorphic function of \( z \) and we have the following uniform (with respect to \( q \)) limits

- \( \lim_{p \to \infty} \gamma = \infty \),
- \( \lim_{p \to \infty} \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial p} = 0 \),
- \( \lim_{p \to \infty} \frac{1}{\gamma} \frac{\partial \varphi}{\partial p} = 0 \).

Then

\[
\lim_{p \to \infty} \frac{1}{F^2} \frac{dF}{dz} = 0.
\]

**Proof:** The result follows directly from the following formula

\[
\left| \frac{F'}{F^2} \right| = (g_{11})^{-\frac{1}{2}} \sqrt{\left( \frac{\partial \gamma}{\partial p} \right)^2 + \gamma^2 \left( \frac{\partial \varphi}{\partial p} \right)^2} = (g_{11})^{-\frac{1}{2}} \sqrt{\left( \frac{\partial \gamma}{\partial p} \right)^2 + \left( \frac{1}{\gamma} \frac{\partial \varphi}{\partial p} \right)^2}.
\]

Thus, under the above hypotheses, for any solution \( W \sim \pm F \). Clearly if the limits in the hypotheses are uniform, then so are the ones in the assertion, so in this case \( W \sim \pm F \) uniformly. In other words, \( F \) and \( -F \) are (uniformly) asymptotic solutions for the Riccati equation \( W' + W^2 = F^2 \).

**4 Exact solutions**

Let \( W(z) = F(z)(1 + \alpha(z)) \) be a solution of \( W' + W^2 = F(z)^2 \), where \( F \) tends to infinity in a domain \( D \). Then \( \alpha(z) \) satisfies the first order ordinary differential equation (2). Thus, in order to obtain solutions \( W \) of \( W' + W^2 = F(z)^2 \) we seek solutions of (2) which tend to zero in \( D \). If we set the nonlinear term in (2) to zero we obtain a linear nonhomogeneous equation

\[
\alpha'_1 + \left( 2F + \frac{F'}{F} \right) \alpha_1 = -\frac{F'}{F}.
\]
We can solve this explicitly for $\alpha_1$ using the variation of parameters formula (3). Now we can substitute $\alpha_1$ for $\alpha$ in the nonlinear term in the original equation (2) to obtain the following linear nonhomogeneous equation

$$\alpha_2' + \left(2F + \frac{F'}{F}\right)\alpha_2 = -\frac{F'}{F} - F\alpha_1^2.$$  

This leads naturally to an iteration scheme described explicitly in the next section, where, assuming $\alpha_0 \equiv 0$, for each $k > 0$ we solve for $\alpha_k$ from the linear nonhomogeneous equation

$$\alpha_k' + \left(2F + \frac{F'}{F}\right)\alpha_k = -\frac{F'}{F} - F\alpha_{k-1}^2.$$  

It will be shown in subsequent sections that the iteration procedure can be performed in such a way that in certain domains $D$, whose shape depends on $F$, as $k \to \infty$ the iterates $\alpha_k$ converge uniformly to a solution $\alpha$ of (2) which tends to zero as $z \to \infty$ in $D$.

5 Iteration

Given a domain $D$ near $\infty$ with a curvilinear system of coordinates $(p, q)$ (see Sec. 3.1), we shall consider a space of continuous functions, denoted by $B$, defined on some sufficiently small $V_\delta = \{z \in D: 1/|p(z)| < \delta\}$ in the previously discussed neighborhood system. The actual value of $\delta$ will be determined later.

**Definition 5.1** Given $\delta > 0$, let $V_\delta = \{z \in D: 1/|p(z)| < \delta\} \subset D$ and define $B$ to be the space of all continuous functions on $V_\delta$ having uniform limit zero as $p \to \infty$. We endow $B$ with the norm $||\alpha|| = \sup_{V_\delta} |\alpha(z)|$.

In $B$ we shall eventually restrict attention to a sequence of iterates of the zero function under the iteration operator $T$ defined below. We shall see that $T$ preserves the zero limit property so $T: B \to B$. The operator $T$ is defined in such a way that each step consists of solving the nonlinear equation (2) by throwing the nonlinear terms to the ‘right-hand side’ and solving the resulting linear nonhomogeneous equation by the method of variation of parameters (see (3)). Thus, if a sequence of iterates of a function under $T$ converges, as we shall see is the case with iterates of the zero function, if $\delta$ is sufficiently small to start with, then the limit function is a solution of the above equation which tends to 0 uniformly in $q$ as $p \to \infty$.

**Definition 5.2** Choose $C = \text{const}$ and $\Phi(z)$ an antiderivative of $F(z)$. Define $T: B \to B$ by

$$(T\alpha)(z) = e^{-\int \left(2F + \frac{F'}{F}\right) d\zeta} \left( C - \int_T \left(\frac{F'}{F} + F\alpha^2\right) e^{\int \left(2F + \frac{F'}{F}\right) d\zeta_1} d\zeta \right)$$
\[
\frac{1}{F} e^{2\Phi} \left( C - \int_\Gamma \left( F' + F^2 \alpha^2 \right) e^{2\Phi} \, d\zeta \right),
\]
where the path of integration \( \Gamma \) is defined as follows:

**Case A:** If in \( D \) we have \( \lim_{p \to \infty} \Re \Phi = -\infty \) uniformly with respect to \( q \), the path of integration \( \Gamma \) is along the segment \( p = \text{const.}, q \in [q(z_0), q(z)] \) and then \( p = [p(z_1), p(z)], q = \text{const.} \).

**Case B:** If in \( D \) we have \( \Re \Phi \) bounded away from \( -\infty \), we choose a fixed \( z_0 \) in \( V \), and for \( z \in V \) let \( z_1 = (p(z), q(z_0)) \). The path of integration \( \Gamma \) is first along the segment \( p = \text{const.}, q = \text{const.} \) and then \( p = [p(z_1), p(z)], q = \text{const.} \).

Note that the criteria are independent of the choice of \( \Phi \). Also since the integrand is holomorphic, once the endpoints are fixed, \( \Gamma \) can be deformed. In the above definition, the particular paths \( \Gamma \) were chosen with easy estimation in mind.

In the case A we need an integrability condition for the existence of the improper integral. This is not an extremely restrictive condition. It is satisfied in many cases and, in fact, we shall show explicitly in Section 7 that this condition is satisfied for one important example.

**Lemma 5.1** Suppose in \( D \) we have \( \lim_{p \to \infty} \Re \Phi = -\infty \) uniformly with respect to \( q \) (i.e. Case A) and \( \alpha \) tends to 0 uniformly with respect to \( q \). If \( e^{2u} ((u_x)^2 + (v_x)^2) \) and \( e^{2u} ((u_{xx})^2 + (v_{xx})^2)^{1/2} \) are integrable to infinity, where \( u \) and \( v \) are the real and imaginary components of \( \Phi \), then we have existence of the improper integrals

\[
\int_\Gamma \left( |F'| + \gamma^2 |\alpha| \right) e^{2\Re \Phi} \, d|\zeta| \quad \text{and} \quad \int_\Gamma \left( F' + F^2 \alpha \right) e^{2\Phi} \, d\zeta,
\]

where \( \Gamma \) is the path \( p \in [p(z), \infty), q = \text{const.} \).

**Proof:** Let \( \Phi = u + iv \). Then \( F = u_x + iv_x \) and \( F' = u_{xx} + iv_{xx} \). Also \( |F|^2 = \gamma^2 = (u_x)^2 + (v_x)^2 \) and \( |F'| = ((u_{xx})^2 + (v_{xx})^2)^{1/2} \). Since \( \alpha \) tends to 0 uniformly with respect to \( q \), we are done, because the integrability conditions imply the existence of the improper integrals

\[
\int_\Gamma |F'| e^{2\Re \Phi} \, d|\zeta| \quad \text{and} \quad \int_\Gamma \gamma^2 e^{2\Re \Phi} \, d|\zeta|.
\]

We shall see that the above two cases correspond to the two cases of l’Hôpital’s rule as applied to the estimate. Case A corresponds to the 0/0 case and, since \( F \to \infty \), case B corresponds to the \( \infty/\infty \) case.

Assume that \( F = \gamma e^{i\varphi} \) is such that ultimately in \( D \), we have \( \cos(\varphi + \psi) \) bounded away from 0. Consider the sequence in \( B \) consisting of \( T^k0, k \geq 1 \). First, we show that all the iterates converge uniformly to 0 as \( p \to \infty \).
Proposition 5.1 Assuming the integrability conditions of the preceding lemma in the Case A, if $\alpha$ tends to 0 uniformly with respect to $q$, so does $T\alpha$.

Proof: Suppose $\alpha \in B$ satisfies $\lim_{p \to \infty} \alpha = 0$ uniformly with respect to $q$. Then, if Re $\Phi$ tends to $-\infty$ (Case A in Definition 5.2)

$$|T\alpha| = \left|\frac{1}{F e^{2\Phi}} \left( C - \int_{\Gamma} \left( F' + F^2 \alpha \right) e^{2\Phi} d\zeta \right) \right|$$

$$\leq \frac{1}{\gamma} |e^{2\Phi}| \left( |C| + \int_{p}^{\infty} \left( |F'| + \gamma^2 |\alpha| \right) |e^{2\Phi}| |d\zeta| \right) = \frac{\mu}{\lambda},$$

where

$$\mu = |C| + \int_{p}^{\infty} \left( |F'| + \gamma^2 |\alpha| \right) e^{2\Re \Phi} |d\zeta| \quad \lambda = \gamma e^{2\Re \Phi}.$$

In this case the improper integral exists by the preceding lemma and integration is along a line $q = \text{const}$, so we have

$$|d\zeta| = \left| \frac{\partial x}{\partial p} + i \frac{\partial y}{\partial p} \right| dp = \sqrt{g_{11}} dp,$$

so

$$\mu = |C| + \int_{p}^{\infty} \left( |F'| + \gamma^2 |\alpha| \right) e^{2\Re \Phi} \sqrt{g_{11}} dp \quad \lambda = \gamma e^{2\Re \Phi}.$$

If Re $\Phi$ tends to $-\infty$ (Case A), we get $\lim_{p \to \infty} \mu = \lim_{p \to \infty} \lambda = 0$, if we assume that $C = 0$ as discussed in the introduction. For $q = \text{const}$ both $\mu$ and $\lambda$ are real valued functions of $p$ and

$$\frac{\mu'}{\lambda} = \frac{\sqrt{g_{11}} \left( |F'| + \gamma^2 |\alpha| \right)}{\gamma' + 2\gamma (\Re \Phi)' \sqrt{g_{11}}} = \frac{\sqrt{g_{11}} \left( |F'| + \gamma^2 |\alpha| \right)}{\gamma' + 2\sqrt{g_{11}} \gamma^2 \cos(\varphi + \psi)}$$

$$= \frac{|F'| + \gamma^2 |\alpha|}{(g_{11})^{-\frac{1}{2}} \gamma' + 2\gamma^2 \cos(\varphi + \psi)},$$

where prime denotes differentiation with respect to $p$. Since $\lim_{p \to \infty} \gamma'/\gamma^2 = 0$, $\lim_{p \to \infty} |F'/F^2| = 0$ and $\lim_{p \to \infty} \alpha = 0$, the uniform limit of $\mu'/\lambda'$ is also zero. As we have seen, application of l'Hôpital’s rule preserves uniformity, so we apply the 0/0 case of the rule to obtain the uniform limit $\lim_{p \to \infty} T\alpha = \lim_{p \to \infty} |T\alpha| = \lim_{p \to \infty} \mu/\lambda = 0$.

If Re $\Phi$ is bounded away from $-\infty$ (Case B), since $\lim_{p \to \infty} \gamma = \infty$, we have $\lim_{p \to \infty} \mu$ $\lim_{p \to \infty} \lambda = \infty$. Here $C$ no longer must be zero. In fact the iteration operator (and its sought-for limit) depends on the choice of $C$ and the choice of $z_0$ in the definition of the path of integration $\Gamma$. The integral along the part of $\Gamma$ with $q = \text{const}$ is treated similarly to the Case A above, except that the limit of integration other than $p(z)$ is $p(z_0)$ instead.
An asymptotic existence theorem in $C^1_\Omega$. The contribution of the integral along the part of $\Gamma$ with $p = p(z_0)$ is bounded by a fixed constant, since $q$ ranges over a compact interval. Since $C$ is arbitrary, the above contribution does not affect the rest of the proof and it follows from the $\infty/\infty$ case of l'Hôpital's rule that uniformly $\lim_D T_\alpha = \lim_{p \to \infty} \mu/\lambda = 0$. ■

6 Convergence

Let $\mathcal{T} \subset \mathcal{B}$ be the set of functions on $V_\delta = \{z \in D: 1/|p(z)| < \delta\}$ consisting of $T^k 0, k \geq 1$. Now we show that if $\delta$ is chosen sufficiently small, then starting from the first iterate $T_0$, we actually get bounds that are uniform with respect to the index of iteration $k$ (and of course with respect to $q$).

Lemma 6.1 Let $\epsilon > 0$. Then $\delta$ can be chosen so small that for all $\alpha \in \mathcal{T}$ on $V_\delta$, we have $||\alpha|| < \epsilon$.

Proof: Suppose $\alpha \in \mathcal{T}$. Then

$$T_0 = \frac{1}{Fe^{2\Phi}} \left( C - \int_{\Gamma} F'e^{2\Phi} d\zeta \right),$$

so

$$||T_\alpha|| = \left| T_0 + \frac{1}{Fe^{2\int_{\Gamma} F'd\zeta}} \int_{\Gamma} \alpha^2 F'e^2 \int_{\Gamma} F'd\zeta dz \right| \leq ||T_0|| + ||\alpha||^2 \left| \frac{\kappa}{\lambda} \right|,$$

where

$$\kappa = \int \gamma^2 e^{2Re\Phi} \sqrt{|g_{11}|} dp, \quad \lambda = \gamma e^{2Re\Phi}.$$

Similarly to the proof for $\mu/\lambda$, we obtain $\kappa/\lambda \to 0$. Here there is no dependence on $\alpha$. Since the zero function has limit 0, so does $T_0$. For sufficiently small $\delta$ we have both $||T_0|| < \frac{\epsilon}{2}$ and $||\kappa/\lambda|| < 1/(2\epsilon)$. By the inductive hypothesis we assume $||\alpha|| < \epsilon$ and for the same $\delta$ obtain

$$||T_\alpha|| < \frac{\epsilon}{2} + \epsilon^2 \frac{1}{2\epsilon} = \epsilon.$$

Finally we show uniform convergence of the sequence $\mathcal{T}$ if we take all our functions on an appropriately small $V$ in the neighborhood system of $D$. Since each iterate tends to 0 uniformly with respect to $q$ as $p \to \infty$, so does the limit function.

Proposition 6.1 If $\delta$ is chosen sufficiently small, then $T$ is a contraction mapping on $\mathcal{T}$ considered as a space of functions on a region $V_\delta = \{z \in D: 1/|p(z)| < \delta\}$. 
Proof: Choose $\delta$ small enough for the conditions of the above lemma to be satisfied for some $\epsilon \in (0, 1)$ and such that $||\kappa/\lambda|| < 1/2$. Suppose $\alpha, \beta \in T$. Then

$$||T\alpha - T\beta|| = \left\| \frac{1}{F^2} \int F d\zeta \int F F^2 \left( \alpha^2 - \beta^2 \right) F^2 d\zeta \right\|$$

$$\leq \left\| \frac{\kappa}{\lambda} \right\| \cdot (||\alpha|| + ||\beta||) \cdot ||\alpha - \beta|| < \frac{1}{2} 2 \epsilon \cdot ||\alpha - \beta|| = \epsilon \cdot ||\alpha - \beta||$$

The preceding results can be combined into an asymptotic existence theorem:

**Theorem 6.1** Suppose $D$ is a region with orthogonal curvilinear coordinates $(p, q)$, $p \in [0, \infty), q \in [0, q_0]$ such that $g_{11} = (\partial x/\partial p)^2 + (\partial y/\partial p)^2$ is bounded away from 0 and $\infty$. Let $F = \gamma e^{i\varphi}$ be a holomorphic function on $D$ satisfying

- $\lim_{p \to \infty} \gamma = \infty$,
- $\lim_{p \to \infty} \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial p} = 0$.
- $\lim_{p \to \infty} \frac{1}{\gamma} \frac{\partial \varphi}{\partial p} = 0$.
- $\cos(\varphi + \psi)$ is bounded away from 0, where $\tan \psi = \frac{\partial y}{\partial p} / \frac{\partial x}{\partial p}$.

Suppose that $\Phi = u + iv$ is an antiderivative of $F$ and if $\text{Re}\Phi \to -\infty$ as $p \to \infty$, then $e^{2u} ((u')^2 + (v')^2)$ and $e^{2u} ((u_{xx})^2 + (v_{xx})^2)^{1/2}$ are integrable to infinity. Then the Riccati equation $W' + W^2 = F(z)^2$ has a solution $W \sim F$ on $D$. In addition, if $\text{Re}\Phi$ is bounded away from $-\infty$, there is a one-parameter family of solutions $W_C \sim -F$.

Note that we may replace $F$ by $-F$ and apply the theorem to obtain solutions (or, depending on the case, one-parameter families of solutions) $W \sim -F$.

7 Applications

Theorem 6.1 can be applied to various situations. Generally a faster growing coefficient function $F$ requires a “smaller” region. In this section we discuss three examples which are of special interest in this work and show that in each case the criteria of the existence Theorem 6.1 are satisfied. In Example 2 we take special care in verifying that the integrability condition of Lemma 5.1 is satisfied in the Case A. For an explanation of the significance
An asymptotic existence theorem in $\mathbb{C}$

of the representation of the canonical flat metric $g_{ij}$ see Lemma 2.1 and the discussion following it.

**Example 1:** In polar coordinates $(p, q) = (\rho, \theta)$ we have

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}.$$ 

Thus, $g_{11} = 1$ and $\psi = \theta$. The theorem applies to functions of bounded argument $\varphi$ in sectors determined from the last condition of the theorem, that $\cos(\varphi + \theta)$ be bounded away from 0. It is easy to show that logarithmic monomials, i.e. functions of the form $\prod_{k=0}^{N}(\log k z)^{m_k}$, where $\log z = z$ and $\log k z = \log(\log k - 1 z)$, satisfy these conditions in sufficiently small sectors. This is the case treated by W. Strodt in [4].

**Example 2:** $e^z$

If $F = e^z$ we choose cartesian coordinates $(p, q) = (x, y)$ in a strip in the right half-plane, e.g. $D = \{z = (x, y): x > x_0, -\pi/2 < y < \pi/2\}$. Here $g_{ij}$ is represented by the identity matrix and $\psi = 0$.

Now we show that $F(z) = e^z$ satisfies the conditions of the theorem in regions properly contained in strips separated by horizontal lines $y = \pi (k + 1/2), k \in \mathbb{Z}$, e.g.

$$\mathcal{D} = \{z = (x, y): x > x_0, y_1 < y < y_2\},$$

where $-\pi/2 < y_1, y_2 < \pi/2$. Note that $\gamma = |F| = |e^z| = e^x$ and $\varphi = \arg F = \arg e^{x+iy} = y$.

- As $p = x \to \infty$, $\gamma = e^x \to \infty$,
- $\gamma^{-2} \frac{\partial \gamma}{\partial p} = \gamma^{-2} \frac{\partial e^x}{\partial x} = e^{-x} \to 0$,
- $\gamma^{-1} \frac{\partial \varphi}{\partial p} = \gamma^{-1} \frac{\partial y}{\partial x} = 0$,
- $\cos(\varphi + \psi) = \cos(y)$ bounded away from 0 by choice of region.

Taking $e^z + C_1$ as an antiderivative of $F$ we see that Re $\Phi = e^x + C_1$ is bounded away from $-\infty$, so we have a one parameter family of solutions asymptotic to $e^z$ and applying the entire argument to $-F$ we have one solution asymptotic to $e^{-z}$.

If we apply the entire argument to $-F$, we need to verify the integrability condition of Lemma 5.1. Let $\Phi = e^z = e^x (\cos y + i \sin y)$. Then $u = e^x \cos y$ and $v = e^x \sin y$, so $u_{xx} = u_x = u = e^x \cos y$ and $v_{xx} = v_x = v = e^x \sin y$. Thus, $(u_x)^2 + (v_x)^2 = e^{2x}$ and $e^{2u} ((u')^2 + (v')^2) = e^{2x} \cos y e^{2x}$ is integrable to infinity. Also $((u_{xx})^2 + (v_{xx})^2)^{1/2} = e^x$ and $e^{2u} ((u_{xx})^2 + (v_{xx})^2)^{1/2} = e^{2x} \cos y e^{2x}$ is integrable to infinity. Thus, the existence theorem is applicable and we have one solution asymptotic to $e^{-z}$. 

Example 3: $e^{e^z}$

Consider a region $D$ in $\mathbb{C}$ bounded by the curves $x = 1, y = 0$ and $y = e^{-\frac{x}{q_0}}$, where $q_0 \in (0, 1)$. Define a system of coordinates $(p, q), p \in [0, \infty), q \in [0, q_0]$ by

$$y = e^{-\frac{p}{q}}, \quad y^2 \left(\ln y - \frac{1}{2}\right) = -x^2 + p^2$$

i.e.

$$p = \sqrt{y^2 \left(\ln y - \frac{1}{2}\right)} + x^2, \quad q = -\frac{x}{\ln y}.$$ 

Taking differentials we obtain

$$dp = \frac{1}{2} \left[ y^2 \left(\ln y - \frac{1}{2}\right) + x^2 \right]^{-\frac{1}{2}} \left[ 2ydy \left(\ln y - \frac{1}{2}\right) + y^2 \frac{1}{y} dy + 2xdx \right] =$$

$$= \left[ y^2 \left(\ln y - \frac{1}{2}\right) + x^2 \right]^{-\frac{1}{2}} (xdx + y \ln y dy),$$

$$dq = -\frac{\ln y dx - x^2 dy}{(\ln y)^2} = \frac{1}{\ln y} \left[ -dx + \frac{x}{y \ln y} dy \right].$$

Now calculate the metric in terms of the new coordinates

$$\langle dp, dp \rangle = \left[ y^2 \left(\ln y - \frac{1}{2}\right) + x^2 \right]^{-1} \left( x^2 + y^2 (\ln y)^2 \right),$$

$$\langle dq, dq \rangle = \frac{1}{(\ln y)^2} \left[ 1 + \frac{x^2}{y^2 (\ln y)^2} \right],$$

$$\langle dp, dq \rangle = \langle dq, dp \rangle = 0,$$

so

$$g^{ij} = \begin{pmatrix}
\frac{x^2 + y^2 (\ln y)^2}{x^2 + y^2 \left(\ln y - \frac{1}{2}\right)} & 0 \\
0 & \frac{1}{(\ln y)^2} \left[ 1 + \frac{x^2}{y^2 (\ln y)^2} \right]
\end{pmatrix},$$

i.e.

$$g_{ij} = \begin{pmatrix}
\frac{x^2 + y^2 \left(\ln y - \frac{1}{2}\right)}{x^2 + y^2 (\ln y)^2} & 0 \\
0 & \left(\ln y\right)^2 \left[ 1 + \frac{x^2}{y^2 (\ln y)^2} \right]^{-1}
\end{pmatrix}.$$
It is easy to check that \( g_{11} \) is bounded away from 0 and \( \infty \). Indeed, as \( p \to \infty \) we have \( y \to 0 \), so \( y^2 \left( \ln y - \frac{1}{2} \right) \to 0 \) and \( y^2(\ln y)^2 \to 0 \). Thus,

\[
g_{11} = \frac{x^2 + y^2 \left( \ln y - \frac{1}{2} \right)}{x^2 + y^2 (\ln y)^2} \to 1 \quad \text{as} \quad p \to \infty.
\]

To estimate \( \psi \) and to calculate the partial derivatives of the argument and norm of \( F \) with respect to \( p \), we resort to implicit differentiation:

\[
\frac{\partial y}{\partial p} = -\frac{y}{q} \frac{\partial x}{\partial p}, \quad 2y \ln y \frac{\partial y}{\partial p} = -2x \frac{\partial x}{\partial p} + 2p.
\]

Substitution of the first equation and \( \ln y = -x/q \) into the second gives

\[
-\ln y \frac{y^2}{q} \frac{\partial x}{\partial p} = -x \frac{\partial x}{\partial p} + p,
\]

so

\[
\frac{\partial x}{\partial p} = \frac{pq^2}{x(q^2 + y^2)}, \quad \frac{\partial y}{\partial p} = -\frac{pqy}{x(q^2 + y^2)}
\]

To calculate \( \psi \) observe that by definition

\[
\tan \psi = \left( \frac{\partial x}{\partial p} \right)^{-1} \frac{\partial y}{\partial p} = -\frac{y}{q}.
\]

Note that \( \partial x/\partial p \to 1 \) as \( p \to \infty \) and since \( g_{11} = (\partial x/\partial p)^2 + (\partial y/\partial p)^2 \to 1 \), we have \( \partial y/\partial p \to 0 \) and \( \tan \psi \to 0 \).

Now let \( F = e^{\gamma y} \). Then \( \gamma = |F| = e^{\Re e^y} = e^{\gamma x} \cos y \) and \( \varphi = \arg F = \Im e^y = e^x \sin y \).

For the partials we get

\[
\frac{\partial \gamma}{\partial p} = \gamma e^x \cos y \frac{\partial x}{\partial p} - \gamma e^x \sin y \frac{\partial y}{\partial p} = \frac{\gamma pq e^x}{x(q^2 + y^2)} [q \cos y + y \sin y],
\]

\[
\frac{\partial \varphi}{\partial p} = e^x \sin y \frac{\partial x}{\partial p} + e^x \cos y \frac{\partial y}{\partial p} = \frac{pq e^x}{x(q^2 + y^2)} [q \sin y - y \cos y].
\]

Thus, as \( p \to \infty \),

\[
\gamma^{-2} \frac{\partial \gamma}{\partial p} = \frac{pq e^x}{\gamma x(q^2 + y^2)} [q \cos y + y \sin y] \to 0,
\]

\[
\gamma^{-1} \frac{\partial \varphi}{\partial p} = \frac{pq e^x}{\gamma x(q^2 + y^2)} [q \sin y - y \cos y] \to 0,
\]
\[ \cos(\varphi + \psi) = \cos \psi \left[ \cos \varphi - \sin \varphi \tan \psi \right] \]
\[ = \pm \left[ 1 + \frac{y^2}{q^2} \right]^{-\frac{1}{2}} \left[ \cos(e^x \sin y) + \sin(e^x \sin y) \frac{y}{q} \right] \to 1, \]

since \( y/q \to 0 \) as \( p \to \infty \) and \( e^x \sin y \leq e^x y \leq e^x e^{-x/q_0} = e^{x(1-1/q_0)} \to 0 \), if \( q_0 < 1 \).

Thus, the conditions of the theorem are satisfied in the region \( D \) defined above:

- As \( p \to \infty \), we have \( x \to \infty \) and \( y \to 0 \), so \( \gamma = e^{e^x \cos y} \to \infty \).
- \( \gamma^{-\frac{1}{2}} \frac{\partial \gamma}{\partial p} \to 0 \),
- \( \gamma^{-1} \frac{\partial \varphi}{\partial p} \to 0 \),
- \( \cos(\varphi + \psi) \) is bounded away from 0.

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References