Differentially transcendental formal power series

Dmitry Gokhman
Division of Mathematics and Statistics
University of Texas at San Antonio

Abstract

We prove that a formal power series in $1/x$, whose coefficients are in a field extension of $\mathbb{Q}$ and are algebraically independent over $\mathbb{Q}$, is differentially transcendental (i.e. not differentially algebraic) over this field extension. This is stated without proof in [2]. This result provides a source of functions analytic at $\infty$ that are not differentially algebraic over $\mathbb{R}$. Such functions are of particular interest, because their germs belong to Hardy fields, but not to the class $E$ of [1] — the intersection of all maximal Hardy fields.

Suppose $F$ is a field extension of the field of rational numbers $\mathbb{Q}$. Let $x$ be an indeterminate and let $u = 1/x$. Let $F[[u]] = F[[1/x]]$ denote the ring of formal power series in $u$ with coefficients in $F$. Then its field of quotients $F((u))$ is a differential field with at least two possible derivations: formal differentiation with respect to $x$ and $u$ denoted by $D_x$ and $D_u$.

Definition 0.1 (see e.g. §VI.1 [7])

Suppose $F$ is an extension field of $K$ and $S \subseteq F$.

(i) $S$ is algebraically dependent over $K$ if a nonzero polynomial in finitely many variables with coefficients in $K$ is annulled by elements of $S$.

(ii) $S$ is a transcendence basis over $K$ means that $S$ is algebraically independent (i.e. not algebraically dependent) over $K$ and is maximal with respect to this property (i.e. $F$ is algebraic over $K(S)$).

(iii) A transcendence basis has unique cardinality (Theorem VI.1.9 [7]) which is called the transcendence degree of $F$ over $K$ and is denoted by $\text{tr.deg.}_KF$.

Definition 0.2 (see e.g. §I.6 [8])

Suppose $F$ is a field, $K$ is a differential field, $F \subseteq K$, and $f \in K$. To say that $f$ is differentially algebraic over $F$ means that $\{f, f', f'', \ldots\}$ is algebraically dependent over $K$, i.e. $f$ is a root of a nonzero differential polynomial with coefficients in $F$.

Proposition 0.1 (Proposition 7.4, [2])

If the set $\{a_i \in F, i = 0, 1, \ldots\}$ is algebraically independent over $\mathbb{Q}$, then

$$f = \sum_{i=0}^{\infty} a_i x^{-i} \in F\left(\left(\frac{1}{x}\right)\right)$$

is not differentially algebraic over $F$ with respect to $D_x$. 

Proof: Suppose \( f \) is a root of a non-zero differential polynomial \( p \) over \( F \) with respect to \( D_x \). Let \( b_j \in F \) \((j = 0, 1, \ldots, 7)\) be the coefficients of \( p \). Then \( f \) is differentially algebraic with respect to \( D_x \) over \( K \), where \( K = \mathbb{Q}(\{b_j\}) \), so
\[
\text{tr.deg}_K K \left( f, D_x f, D_x^2 f, \ldots \right) < \infty.
\]
Since \( \text{tr.deg}_Q K < \infty \), we have
\[
\text{tr.deg}_Q K \left( f, D_x f, D_x^2 f, \ldots \right) =
\]
\[
\text{tr.deg}_Q K + \text{tr.deg}_K K \left( f, D_x f, D_x^2 f, \ldots \right) < \infty.
\]
Therefore
\[
\text{tr.deg}_Q Q \left( f, D_x f, D_x^2 f, \ldots \right) =
\]
\[
\text{tr.deg}_Q K \left( f, D_x f, D_x^2 f, \ldots \right) - \text{tr.deg}_Q Q(f, D_x f, D_x^2 f, \ldots) K \left( f, D_x f, D_x^2 f, \ldots \right) < \infty.
\]
Note that
\[
D_x f = D_u f \left( -\frac{1}{x^2} \right) = D_u F \left( -u^2 \right),
\]
\[
D_x^2 f = \left( D_u^2 f \left( -u^2 \right) + D_u f (-2u) \right) \left( -u^2 \right)
\]
and so on, so \( Q(x, f, D_x f, D_x^2 f, \ldots) = Q(u, f, D_u f, D_u^2 f, \ldots) \).

Now since \( \text{tr.deg}_Q Q(u) = 1 \) and \( \text{tr.deg}_Q Q(f, D_x f, D_x^2 f, \ldots) < \infty \), we have
\( \text{tr.deg}_Q Q(u, f, D_u f, D_u^2 f, \ldots) < \infty \) and \( \text{tr.deg}_Q Q(f, D_u f, D_u^2 f, \ldots) < \infty \). Therefore, \( f \) satisfies some non-zero differential polynomial \( q \) over \( Q \) with respect to \( D_u \). Since the field \( Q \) is infinite, there exist \( t_k \in Q \) \((k = 0, 1, \ldots, \infty)\) such that \( q(t_0, t_1, \ldots, t_k) \neq 0 \). Let
\[
g(u) = \sum_{k=0}^{\infty} \frac{t_k}{k!} u^k.
\]
Then \( q(g, D_u g, \ldots) \neq 0 \), because it is non-zero when evaluated at \( u = 0 \) (it equals \( q(\{t_k\}) \)).

If \( h \) is a formal sum \( \sum_{i=0}^{\infty} y_i u^i \) with \( y_i \) indeterminate, then \( q(h, D_u h, \ldots) \) is a power series in \( u \), whose coefficients are polynomials in \( y_i \) over \( \mathbb{Z} \). Since evaluating \( y_i \) at \( (t_0, t_1, \ldots, \infty) \) produces a non-zero answer, one of the above coefficients is a non-zero polynomial in \( y_i \) over \( Q \). Let \( r \) denote this polynomial. Evaluating \( y_i \) at \( a_i \) gives \( q(f, D_u f, \ldots) = 0 \), so, in particular, \( r(a_i) = 0 \), which contradicts the assumption that \( a_i \) are algebraically independent over \( Q \). \( \blacksquare \)

The value of this result lies in providing a class of functions analytic at \( \infty \) that are differentially transcendental over \( \mathbb{R} \). The germs of such functions necessarily belong to Hardy
Differentially transcendental formal power series

fields (see the definition below), but not to Boshernitzan’s class $E$ [1] — the intersection of all maximal Hardy fields. \(^1\)

**Definition 0.3** (see [4]) A differential field of continuous germs \(^2\) of real functions at $+\infty$, where the derivation is ordinary differentiation, is called a Hardy field.

**Corollary 1** Suppose $f$ is a real function that is analytic at $\infty$ and the set of coefficients of its Taylor series at $\infty$ is algebraically independent over $\mathbb{Q}$. Then

(i) $f$ is differentially transcendental over $\mathbb{R}$,

(ii) $\mathbb{R}(f, f', ...) $ is a Hardy field,

(iii) $f \notin E$, where $E$ is the intersection of all maximal Hardy fields.

**Proof:** (i) is a special case of Proposition 0.1 with $F = \mathbb{R}$ and the series actually convergent. (ii) is a consequence of the fact that the zeros of an analytic function are necessarily isolated and, thus, cannot have $\infty$ as an accumulation point. This means that every nonzero element of $\mathbb{R} [f, f', ...]$, i.e. a nonzero differential polynomial of $f$, is invertible in a (punctured) neighborhood of $\infty$ (see Theorem 7.1 [1]). (iii) is a consequence of the fact that $E$ is a differentially algebraic extension of $\mathbb{R}$ (Theorem 14.4 [2]). ■

The few other known examples of classes of functions satisfying the conditions of Corollary 1 include

(i) Euler’s $\Gamma$-function, which is not differentially algebraic over $\mathbb{R}$ by Hölder’s theorem [6] and generates a Hardy field [9];

(ii) Functions represented by a Dirichlet series

$$
\sum_{n=1}^{\infty} \frac{a_n}{n^x},
$$

where $a_n \in \mathbb{R}$ are subpolynomial in $n$ and the set of all prime divisors of $n$ in the support of $a_n$ \(^3\) is infinite, e.g. the Riemann $\zeta$-function on a positive half-line [9];

(iii) The function

$$
\sum_{n=1}^{\infty} \frac{1}{e_n(x)},
$$

where $e_1(x) = e^x$ and $e_n(x) = e^{e_{n-1}(x)}$ for $n > 1$ [9];

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\(^1\) $E$ is an extension of Hardy’s class $L$ of logarithmico-exponential functions [5] and is the maximal scale for functions in Hardy fields (functions of regular growth).

\(^2\) A germ is an equivalence class of functions, where two functions are equivalent exactly when they agree in a neighborhood of the point of interest, in our case $+\infty$.

\(^3\) The support of $a_n$ is the set of all $n$ such that $a_n \neq 0$. 
(iv) Certain fractional iterates of $e^x$ [3];

(v) Certain ultimately $^4\mathcal{C}^\infty$ transexponential solutions of two difference equations: $f(x + 1) = e^{f(x)}$ and $f(x + 1) = e^{f(x)} - 1$ [3].

References


\(^4\) A property is said to hold ultimately exactly when it holds in a neighborhood of $+\infty$. 