A new approach to turbulent internal flows with high Reynolds numbers

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The authors have developed a method for fast and extremely accurate computation of axisymmetric or two-dimensional internal turbulent flows with Reynolds number between $10^7$ and $10^8$. The main ingredients are a modification of the Galerkin method based on splining (to overcome difficulties of the standard Galerkin method at the boundary) and a curvilinear orthogonal coordinate system following the boundary and approximately following the flow lines. I will describe the two techniques and present some applications.


A. Gokhman, D. Gokhman, Boundary element method for axisymmetric flow, In preparation
Boundary Element Method for Axisymmetric Flow

Governing equation: \( \nabla^2 \varphi = 0 \)

\[
\frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial r^2} = 0, \quad v_r = \frac{\partial \varphi}{\partial r}, \quad v_z = \frac{\partial \varphi}{\partial z}, \quad v_\theta = 0
\]

Boundary conditions:

(a) Radial with distance between the planes \( b_0 \): \( v_z < v_r \sim \pm \frac{Q}{2\pi rb_0} \) as \( r \to \infty \)

(b) Cylinder with radius \( r_0 \): \( v_r \to 0, v_z \to \pm \frac{Q}{\pi r_0^2} \) as \( z \to \pm \infty \),

(c) Cone with vertex \((0, z_0)\) and generatrix \( r = \tan \alpha (z - z_0)\):

\[
v(r, z) \sim \pm \frac{Q}{2\pi(1 - \cos \alpha) |\mathbf{n}|^2} \mathbf{n}, \text{ as } |\mathbf{n}| \to \infty, \text{ where } \mathbf{n} = (r, z - z_0).
\]

\[
\gamma = \frac{dr}{d\ell} \frac{u_r}{u_z} + \frac{dz}{d\ell} \frac{u_z}{u_z}
\]

\[
u_r = \frac{1}{4\pi} \int_L \gamma(\ell)r(\ell)(z_c - z(\ell)) \int_0^{2\pi} \frac{\sin \theta}{(r(\ell)^2 + r_c^2 - 2r_c r(\ell) \sin \theta + (z_c - z(\ell))^2)^{\frac{3}{2}}} d\theta d\ell
\]

\[
u_z = \frac{1}{4\pi} \int_L \gamma(\ell)r(\ell) \int_0^{2\pi} \frac{r(\ell) - r_c \sin \theta}{(r(\ell)^2 + r_c^2 - 2r_c r(\ell) \sin \theta + (z_c - z(\ell))^2)^{\frac{3}{2}}} d\theta d\ell
\]

(a) Radial: \( \gamma \sim \pm \frac{Q}{2\pi b_0 r} \) as \( r \to \infty \) with opposite signs on the two planes.

(b) Cylinder: \( \gamma \to \pm \frac{Q}{\pi r_0^2} \) as \( z \to \pm \infty \)

(c) Cone: \( \gamma \sim \pm \frac{Q}{2\pi(1 - \cos \alpha) |\mathbf{n}|^2} \) as \( |\mathbf{n}| \to \infty \), where \( \mathbf{n} = (r, z - z_0) \)
We can compute the integrals with respect to $d\ell$ exactly using the following table integrals

\[
T_n = \int \frac{\ell^n \, d\ell}{(\ell^2 + b\ell + a)^{\frac{3}{2}}} \quad [n = 0, 1, -1]
\]

\[
T_0 = \frac{2(2\ell + b)}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}}
\]

\[
T_1 = -\frac{2(2a + b\ell)}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}}
\]

\[
T_2 = -2\frac{(3a - b^2)\ell - ab}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} + \ln \left( \frac{(\ell^2 + b\ell + a)^{\frac{1}{2}}}{2} \right) + 2 + b + \ell
\]

\[
T_3 = \frac{(4a - b^2)\ell^2 + b(10a - 3b^2)\ell + a(8a - 3b^2)}{(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} - \frac{3b}{2} \ln \left( \frac{(\ell^2 + b\ell + a)^{\frac{1}{2}}}{2} \right) + 2 + b + \ell
\]

\[
T_{-1} = -\frac{2(b\ell - 2a + b^2)}{a(4a - b^2)(\ell^2 + b\ell + a)^{\frac{3}{2}}} - \frac{1}{a^{\frac{3}{2}}} \ln \left( \frac{2a + b\ell + 2a^{\frac{1}{2}}(\ell^2 + b\ell + a)^{\frac{1}{2}}}{\ell} \right)
\]

and their definite forms

\[
T^*_n = \int_{\ell_0}^{\infty} \frac{\ell^n \, d\ell}{(\ell^2 + b\ell + a)^{\frac{3}{2}}} = \lim_{\ell \to \infty} T_n(\ell) - T_n(\ell_0) \quad [n = 0, 1, -1]
\]

\[
T^*_0 = \frac{2}{4a - b^2} \left( 2 - \frac{2\ell_0 + b}{(\ell_0^2 + b\ell_0 + a)^{\frac{3}{2}}} \right)
\]

\[
T^*_1 = \frac{2}{4a - b^2} \left( \frac{2a + b\ell_0}{(\ell_0^2 + b\ell_0 + a)^{\frac{3}{2}}} - b \right)
\]

\[
T^*_{-1} = \frac{2}{a(4a - b^2)} \left( \frac{b\ell_0 - 2a + b^2}{(\ell_0^2 + b\ell_0 + a)^{\frac{3}{2}}} - b \right) + \frac{1}{a^{\frac{3}{2}}} \ln \left( \frac{2a + b\ell_0 + 2a^{\frac{1}{2}}(\ell_0^2 + b\ell_0 + a)^{\frac{1}{2}}}{(b + 2a^{\frac{1}{2}})\ell_0} \right)
\]

Radial: $r(\ell) = \ell$, $\ell > \ell_0$, $z = z_0$

\[
a = r_c^2 + (z_c - z_0)^2, \quad b = -2r_c \sin \theta
\]

\[
u_r = \pm \frac{Q}{8\pi^2 b_0} \int_0^{2\pi} \int_{\ell_0}^{\infty} \frac{(z_c - z_0) \sin \theta}{(\ell^2 + r_c^2 - 2r_c\ell \sin \theta + (z_c - z_0)^2)^{\frac{3}{2}}} \, d\ell \, d\theta
\]

\[
= \pm \frac{Q}{8\pi^2 b_0} \int_0^{2\pi} \frac{(z_c - z_0) \sin \theta \, T_0^* \, d\theta}{\ell - r_c \sin \theta}
\]

\[
= \pm \frac{Q}{8\pi^2 b_0} \int_0^{2\pi} \frac{(z_c - z_0) \sin \theta \, d\theta}{(\ell^2 + r_c^2 - 2r_c\ell \sin \theta + (z_c - z_0)^2)^{\frac{3}{2}}}
\]

\[
u_z = \pm \frac{Q}{8\pi^2 b_0} \int_0^{2\pi} \int_{\ell_0}^{\infty} \frac{\ell - r_c \sin \theta}{(\ell^2 + r_c^2 - 2r_c\ell \sin \theta + (z_c - z_0)^2)^{\frac{3}{2}}} \, d\ell \, d\theta
\]

\[
= \pm \int_0^{2\pi} \frac{T_1^* - r_c \sin \theta \, T_0^* \, d\theta}{\ell^2 + r_c^2 - 2r_c\ell \sin \theta + (z_c - z_0)^2} + \frac{d\theta}{(\ell_0^2 + r_c^2 - 2r_c\ell_0 \sin \theta + (z_c - z_0)^2)^{\frac{3}{2}}}
\]
Finite elements with closed form Galerkin integrals

To implement the finite element scheme we will use the following master elements in dimensionless variables $\xi$ and $\eta$: 4-node element with bilinear tensor Lagrangian interpolating shape functions for pressure and 9-node element for biquadratic tensor Lagrange interpolating shape functions for velocity as illustrated in the following figure and tables.

Master elements in dimensionless variables $\xi$ and $\eta$

<table>
<thead>
<tr>
<th>Deg.</th>
<th>Nodes</th>
<th>Polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{-1, 1}</td>
<td>$\ell_{11}(x) = \frac{1}{2} (1 - x)$ $\ell_{12}(x) = \frac{1}{2} (1 + x)$</td>
</tr>
<tr>
<td>2</td>
<td>{-1, 0, 1}</td>
<td>$\ell_{21}(x) = \frac{1}{2} x (x - 1)$ $\ell_{22}(x) = 1 - x^2$ $\ell_{23}(x) = \frac{1}{2} x (x + 1)$</td>
</tr>
</tbody>
</table>

Lagrange interpolating polynomials

$\hat{\varphi}_1(\xi, \eta) = \ell_{11}(\xi)\ell_{11}(\eta)$ $\hat{\varphi}_2(\xi, \eta) = \ell_{12}(\xi)\ell_{11}(\eta)$ $\hat{\varphi}_3(\xi, \eta) = \ell_{12}(\xi)\ell_{12}(\eta)$ $\hat{\varphi}_4(\xi, \eta) = \ell_{11}(\xi)\ell_{12}(\eta)$

Bilinear tensor Lagrange shape functions for the 4-node element

$\hat{\psi}_1(\xi, \eta) = \ell_{21}(\xi)\ell_{21}(\eta)$ $\hat{\psi}_2(\xi, \eta) = \ell_{23}(\xi)\ell_{21}(\eta)$ $\hat{\psi}_3(\xi, \eta) = \ell_{23}(\xi)\ell_{23}(\eta)$ $\hat{\psi}_4(\xi, \eta) = \ell_{21}(\xi)\ell_{23}(\eta)$ $\hat{\psi}_5(\xi, \eta) = \ell_{22}(\xi)\ell_{21}(\eta)$ $\hat{\psi}_6(\xi, \eta) = \ell_{22}(\xi)\ell_{23}(\eta)$ $\hat{\psi}_7(\xi, \eta) = \ell_{22}(\xi)\ell_{22}(\eta)$ $\hat{\psi}_8(\xi, \eta) = \ell_{21}(\xi)\ell_{22}(\eta)$ $\hat{\psi}_9(\xi, \eta) = \ell_{22}(\xi)\ell_{22}(\eta)$

Biquadratic tensor Lagrange shape functions for the 9-node element

$\bar{p}_i = p_{i-1} + \Delta p$ $\bar{p}_{i+1} = p_i + \Delta p$

$\bar{q}_i = q_{i-1} + \Delta q$ $\bar{q}_{i+1} = q_i + \Delta q$

$\xi = \frac{p - \bar{p}_i}{\Delta \bar{p}}$ $(-1 \leq \xi \leq 1)$ $\eta = \frac{q - \bar{q}_i}{\Delta \bar{q}}$ $(-1 \leq \eta \leq 1)$

The relationship between $p, q$ and $\xi, \eta$
The governing equations for incompressible steady two-dimensional flow are the conservation of mass (continuity) equation
\[ \nabla \cdot \mathbf{v} = 0 \]
and the balance of momentum (Navier-Stokes) equation
\[ \frac{1}{2} \nabla \left( |\mathbf{v}|^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{F} - \frac{1}{\rho} \nabla P + (\nu + \tau) \nabla^2 \mathbf{v}, \]
where
- \( \rho \) mass density
- \( \mathbf{v} \) velocity
- \( \mathbf{F} \) body force density
- \( P \) pressure
- \( \nu \) Newtonian kinematic viscosity
- \( \tau \) turbulent kinematic viscosity

The governing equations written in curvilinear orthogonal coordinates \( p \) and \( q \) in the two-dimensional are
\[
\frac{1}{h_p} \left[ \frac{\partial v_p}{\partial p} + \frac{v_q}{h_p h_q} \frac{\partial v_q}{\partial q} \right] - \frac{v_q}{h_p h_q} \left[ \frac{\partial (h_q u_q)}{\partial p} - \frac{\partial (h_p u_p)}{\partial q} \right] = F_p - \frac{1}{\rho h_p} \frac{\partial P}{\partial p} \\
+ \frac{\nu + \tau}{h_p h_q} \left[ \frac{\partial}{\partial p} \left( \frac{h_q}{h_p} \frac{\partial v_p}{\partial p} \right) + \frac{\partial}{\partial q} \left( \frac{h_p}{h_q} \frac{\partial v_p}{\partial q} \right) \right],
\]
\[
\frac{1}{h_q} \left[ \frac{\partial v_q}{\partial q} + \frac{v_p}{h_p h_q} \frac{\partial v_p}{\partial p} \right] + \frac{v_p}{h_p h_q} \left[ \frac{\partial (h_q u_q)}{\partial q} - \frac{\partial (h_p u_p)}{\partial p} \right] = F_q - \frac{1}{\rho h_q} \frac{\partial P}{\partial q} \\
+ \frac{\nu + \tau}{h_p h_q} \left[ \frac{\partial}{\partial q} \left( \frac{h_q}{h_p} \frac{\partial v_q}{\partial q} \right) + \frac{\partial}{\partial p} \left( \frac{h_p}{h_q} \frac{\partial v_q}{\partial p} \right) \right],
\]
where \( v_p \) and \( v_q \) are the components of \( \mathbf{v} \), \( F_p \) and \( F_q \) are the components of \( \mathbf{F} \), and \( h_p \) and \( h_q \) are the metrical coefficients (Lamé functions).
\[ \bar{P} = \sum_{k=1}^{4} \varphi_k P_k \quad \bar{v}_p = \sum_{k=1}^{9} \hat{\psi}_k v_{pk} \quad \bar{v}_q = \sum_{k=1}^{9} \hat{\psi}_k v_{qk} \]

Continuity equation residual:

\[ R_c(v_{pk},v_{qk}) = \sum_{k} \left( \frac{\partial h_q}{\partial p} \psi_k + h_q \frac{\partial \psi_k}{\partial p} \right) v_{pk} + \left( \frac{\partial h_p}{\partial q} \psi_k + h_p \frac{\partial \psi_k}{\partial q} \right) v_{qk} \]

\[ \langle R_c, \varphi_j \rangle = \sum_k \left( \left( \frac{\partial h_q}{\partial p} \psi_k, \varphi_j \right) + \left( h_q \frac{\partial \psi_k}{\partial p}, \varphi_j \right) \right) v_{pk} + \left( \left( \frac{\partial h_p}{\partial q} \psi_k, \varphi_j \right) + \left( h_p \frac{\partial \psi_k}{\partial q}, \varphi_j \right) \right) v_{qk} \]

where

\[ \langle f, g \rangle = \int \int f g h_p h_q \, dp \, dq \]

Spline, e.g.:

\[ \frac{\partial h_q}{\partial p} h_p h_q = s(\lambda, \mu) = \sum_{i=1}^{12} s_i \bar{\omega}_i = s(0, 0) \bar{\omega}_1 + s(1, 0) \bar{\omega}_2 + s(1, 1) \bar{\omega}_3 + s(0, 1) \bar{\omega}_4 \]

\[ + \frac{\partial s}{\partial p} (0, 0) \Delta p \bar{\omega}_5 + \frac{\partial s}{\partial p} (1, 0) \Delta p \bar{\omega}_6 + \frac{\partial s}{\partial p} (1, 1) \Delta p \bar{\omega}_7 + \frac{\partial s}{\partial p} (0, 1) \Delta p \bar{\omega}_8 \]

\[ + \frac{\partial s}{\partial q} (0, 0) \Delta q \bar{\omega}_9 + \frac{\partial s}{\partial q} (1, 0) \Delta q \bar{\omega}_{10} + \frac{\partial s}{\partial q} (1, 1) \Delta q \bar{\omega}_{11} + \frac{\partial s}{\partial q} (0, 1) \Delta q \bar{\omega}_{12} \]

where

\[ \bar{\omega}_1(\lambda, \mu) = H_{01}(\lambda) H_{01}(\mu) \quad \bar{\omega}_2(\lambda, \mu) = H_{02}(\lambda) H_{01}(\mu) \quad \bar{\omega}_3(\lambda, \mu) = H_{02}(\lambda) H_{02}(\mu) \]

\[ \bar{\omega}_4(\lambda, \mu) = H_{01}(\lambda) H_{02}(\mu) \quad \bar{\omega}_5(\lambda, \mu) = H_{11}(\lambda) H_{01}(\mu) \quad \bar{\omega}_6(\lambda, \mu) = H_{12}(\lambda) H_{01}(\mu) \]

\[ \bar{\omega}_7(\lambda, \mu) = H_{12}(\lambda) H_{02}(\mu) \quad \bar{\omega}_8(\lambda, \mu) = H_{11}(\lambda) H_{02}(\mu) \quad \bar{\omega}_9(\lambda, \mu) = H_{01}(\lambda) H_{11}(\mu) \]

\[ \bar{\omega}_{10}(\lambda, \mu) = H_{02}(\lambda) H_{11}(\mu) \quad \bar{\omega}_{11}(\lambda, \mu) = H_{02}(\lambda) H_{12}(\mu) \quad \bar{\omega}_{12}(\lambda, \mu) = H_{01}(\lambda) H_{12}(\mu) \]

where \( \lambda \) and \( \mu \) are shifts of \( \xi \) and \( \eta \) and \( H \)'s are Hermite interpolant polynomials:

\[ H_{01}(x) = 1 - 3x^2 + 2x^3 \quad H_{02}(x) = 3x^2 - 2x^3 \]

\[ H_{11}(x) = x - 2x^2 + x^3 \quad H_{12}(x) = -x^2 + x^3 \]

Top left: \( \lambda = \xi + 1 \) and \( \mu = \eta \)

\[ \int \int \bar{\omega}_7 \hat{\psi}_2 \hat{\phi}_1 \, dp \, dq = \frac{\Delta p \Delta q}{16} \int_0^1 \int_{-1}^0 \left( -\xi^6 - 2\xi^5 + 2\xi^3 + \xi^2 \right) \left( 2\eta^6 - 7\eta^5 + 8\eta^4 - 3\eta^3 \right) \, d\xi \, d\eta = -\frac{13\Delta p \Delta q}{282240} \]
The Galerkin–Gokhman method

\[ v_{pg} = \sum \psi_k v_{pk} \quad v_{qg} = \sum \psi_k v_{qk} \quad \Delta v_p = v_{ps} - v_{pg} \quad \Delta v_q = v_{qs} - v_{qg} \]

where \( v_{ps} \) and \( v_{qs} \) are splined.

\[ \dot{v} + \Delta \dot{v} = \text{eqs.} \]

Modified continuity eq:

\[ \frac{\partial (h_q v_p)}{\partial p} + \frac{\partial (h_p v_q)}{\partial q} = -\frac{\partial (h_q \Delta v_p)}{\partial p} - \frac{\partial (h_p \Delta v_q)}{\partial q} \]

Fig. 3. Parametric study of accuracy of solutions for systems describing fully developed flow in a pipe using Galerkin and Galerkin-Gokhman methods at \( Re = 10^7 \).