1 Project Description

The lectures will be concentrated on studying finite Morse index solutions of nonlinear elliptic partial differential equations on unbounded domains. Finite Morse index solutions are "not too unstable" and thus are more likely to be physically relevant. These results are then used to study problems on bounded domains (with boundary conditions). Firstly, they are used to study the behaviour of the large positive solutions on branches of solutions of nonlinear elliptic equations if the nonlinearity grows rather rapidly. In particular, we show that the branch has infinitely many turning points or bifurcation points. Secondly, we use similar ideas to obtain a good understanding of stable or "not too unstable" solutions when the diffusion is small. It is also intended to include some basic material on elliptic partial differential equations and some subsidiary lectures on the important related problem of looking for solutions with sharp peaks when the diffusion is small.

Lecture 1. An introduction to the course and a short reminder on weak derivatives, weak solutions, estimates, sub and super solutions, and Harnack inequalities: These are the technical tools which are extensively used in the course. It is intended to have a supplementary session to discuss these ideas more carefully for participants less prepared in the area.

Lecture 2. Monotone and linearized stable solutions on R^n: In this lecture, we discuss and modify ideas of Ambrosio and Cabre to show that in low dimensions, linearized stable bounded solutions of \(-\Delta u = f(u)\) on \(R^n\) are monotone solutions of one variable (after a rotation of coordinates). Here a solution is said to be linearized stable if \(J(h) \geq 0\) for all smooth \(h\) of compact support in \(R^n\), where \(J\) is the natural quadratic form corresponding to the linearization of our equation at \(u\). We also discuss open problems, and the work of Savin.

Lecture 3. We consider the half space case with a Dirichlet boundary condition and show how our ideas can be combined with moving plane ideas when the solutions are assumed to be positive and decaying in one direction.

Lecture 4. We use the theories in Lectures 2 and 3 to study bounded finite Morse index solutions on \(R^n\). Here a solution \(u\) of \(-\Delta u = f(u)\) on \(R^n\) is said to be of finite Morse index if it is not too unstable; more formally, if the negative spectrum of the self-adjoint linear operator \(-\Delta - f'(u)I\) consists of
a finite number of points each of finite multiplicity.

Lectures 5-6. We use the theory of lectures 2, blow-up methods and older methods for nonlinear elliptic equations to show that in low dimensions, we can classify completely the uniformly bounded stable positive solutions of

\[-e^2 \Delta u = f(u) \text{ in } W, \quad u = 0 \text{ on } \partial W\]

for \(e\) small if \(W\) is a smooth bounded domain in \(\mathbb{R}^n\) and \(f : \mathbb{R} \to \mathbb{R}\) is \(C^1\) with \(f(0) \geq 0\). In the process, we will give an introduction to blow-up methods, discuss related results (including other boundary conditions), open problems, and the case where we do not assume positivity. We also discuss why the problem must be far more complicated if \(e\) is not small (because of domain variation ideas).

Lecture 7. We show that we can use Lecture 4 and similar ideas to obtain for small \(e\) a very good understanding of the uniformly bounded positive solutions whose Morse index is bounded for a very large class of \(f\)s (where the Morse index is the number of negative eigenvalues of the linearization of (*) at a solution \(u\) counting multiplicity). In particular, we find that these solutions are stable solutions with a finite number of sharp peaks superimposed. In companion lectures, a short introduction to the use of finite-dimensional reduction methods to study these solutions will be given (by another lecturer).

Lectures 8-9. We modify the ideas of Lectures 2-4 to prove that if \(1 < p < n(n - 4), p < \infty\) if \(n = 2, 3, 4\), the equation \(-\Delta u = |u|^{p-1}u\) has no bounded nontrivial bounded finite Morse index solution on \(\mathbb{R}^n\). Here the idea is to first prove the result for linearized stable solutions and then use scaling and blow-up techniques, together with more classical techniques.

We discuss briefly some applications to supercritical problems on bounded domains and similar results for solutions of \(-\Delta u = e^u\) if \(n = 3\) (where the solutions may not be bounded).

Lecture 10. In the first part of this lecture we introduce the ideas of analytic bifurcation theory; in particular, results which only hold in the analytic case. We then use this and the theory of Lectures 8-9 to give significant generalizations of the classical results of Joseph and Lungren [12] (following earlier work of Chandrasekhar and Gelfand) on infinitely many bifurcations for the
branch of positive solutions of

$$-\Delta u = \lambda f(u) \text{ in } W, \ u = 0 \text{ on } \partial W$$

when $f$ grows supercritically at infinity (the original results were only for $W$ a ball and very particular $f$’s and used ordinary differential equations techniques).