Riemann Sum and Approximation of Integrals

Uploading the Student Package:
```maple
> with(student);
[D, Diff, Doubleint, Int, Limit, Lineint, Product, Sum, Tripleint, changevar, completesquare, distance, equate, integrand, intercept, intparts, leftbox, leftsum, makeproc, middlebox, middlesum, midpoint, powsubs, rightbox, rightsum, showtangent, simpson, slope, summand, trapezoid]
```

Enter a function into Maple
```maple
> f := x -> root[3](1 + 3 \cdot x^5 - x^2);
f := x -> root[3](1 + 3 \cdot x^5 - x^2)
```

Plot the function over [2,5]
```maple
> plot(f(x), x = 2 .. 5);
```

We plot the Riemann approximation using ten rectangles built by left end points.
```maple
> leftbox(f(x), x = 2 .. 5, 10);
```
We plot the Riemann approximation using right end points.

> rightbox( f(x), x = 2 .. 5, 10 );
We plot the Riemann approximation using middle points.

\[ \text{middlebox}(f(x), x = 2 \ldots 5, 10); \]
Let's compute both left and right sums

\>
> evalf (leftsum (f(x), x = 2 ..5, 10));
> evalf (rightsum (f(x), x = 2 ..5, 10));

Notice that, since the function is increasing, the leftsums are always less than the rightsums and the actual area (or definite integral of f(x)) is between the values of these sums. With ten rectangles, we see that the difference of these sums is about 5 so that they approximate the integral by an error of 5. To get a better approximation, we have to increase the number of rectangles.

Let's compute the left and right sums with different numbers of rectangles (we call the number by N).
and their differences:

\[ N := 40; L := \text{evalf(leftsum}(f(x), x = 2 .. 5, N)); R := \text{evalf(rightsum}(f(x), x = 2 .. 5, N)); \]
\[ \text{err} := R - L; \]
\[ N := 40 \]
\[ L := 35.39435523 \]
\[ R := 36.63464171 \]
\[ \text{err} := 1.24028648 \]

(7)

Try again with bigger N:

\[ N := 100; L := \text{evalf(leftsum}(f(x), x = 2 .. 5, N)); R := \text{evalf(rightsum}(f(x), x = 2 .. 5, N)); \]
\[ \text{err} := R - L; \]
\[ N := 100 \]
\[ L := 35.76518088 \]
\[ R := 36.26129550 \]
\[ \text{err} := 0.49611462 \]

(8)

\[ N := 200; L := \text{evalf(leftsum}(f(x), x = 2 .. 5, N)); R := \text{evalf(rightsum}(f(x), x = 2 .. 5, N)); \]
\[ \text{err} := R - L; \]
\[ N := 200 \]
\[ L := 35.88902950 \]
\[ R := 36.13708680 \]
\[ \text{err} := 0.24805730 \]

(9)

\[ N := 500; L := \text{evalf(leftsum}(f(x), x = 2 .. 5, N)); R := \text{evalf(rightsum}(f(x), x = 2 .. 5, N)); \]
\[ \text{err} := R - L; \]
\[ N := 500 \]
\[ L := 35.96339628 \]
\[ R := 36.06261920 \]
\[ \text{err} := 0.09922292 \]

(10)

\[ N := 10000; L := \text{evalf(leftsum}(f(x), x = 2 .. 5, N)); R := \text{evalf(rightsum}(f(x), x = 2 .. 5, N)); \]
\[ \text{err} := R - L; \]
\[ N := 10000 \]
\[ L := 36.01051758 \]
\[ R := 36.01547874 \]
\[ \text{err} := 0.00496116 \]

(11)

So, with 10000 (!) rectangle, we can say that the definite integral

\[ \int_{2}^{5} \left( 1 + 3 x^5 - x^2 \right)^{1/3} \, dx \]

is about 36.01051758 (using the value of the left sum) with an error of 0.00496116!

We can use Maple to compute this integral:

\[ \text{int}(f(x), x = 2 .. 5); \]
\[ \int_{2}^{5} \left( 1 + 3 x^5 - x^2 \right)^{1/3} \, dx \]

(12)

\[ \text{evalf} \%; \]
\[ 36.01299814 \]

(14)
We can see that the left sum (36.01275009) approximates the integral (~36.01299814) within an error of 0.00049611. Here I used 100,000 rectangles and it took 2 mins on my laptop!