Complex Variables midterm II

Note: The curves described below (except the one in #5) are all oriented counterclockwise.

1. Let $\gamma$ be the perimeter of the squares centered at $i$ with sides length being 2. Compute $\int_{\gamma} f(z)dz$ where

   \begin{align*}
   a) \quad f(z) &= z^2 + e^z \quad (z - 1)^3, \\
   b) \quad f(z) &= \frac{z^3 + iz + 1}{(z - 2i)^2}.
   \end{align*}

2. Let $a$ be a real number and $|a| < 1$. Explicitly write down the definition of the integral $\int_{\gamma} \frac{dz}{z-a}$ and show that

   \[ \int_0^{2\pi} \frac{1 - a \cos(t)}{1 - 2a \cos(t) + a^2} dt = 2\pi. \]

   What happens if $|a| > 1$?

3. Let $f$ be analytic in $\Omega$ and $\gamma$ be a closed path whose interior containing 2 points $z_1, z_2$. Show that

   \[ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_1)(z-z_2)} dz = \frac{f(z_1)}{(z_1-z_2)} + \frac{f(z_2)}{(z_2-z_1)}. \]

   How do you generalize this formula?

   Let $\gamma$ be the square with corners at $\pm 100$ and $\pm 100i$. Using the above result to compute

   \begin{align*}
   a) \quad \int_{\gamma} \frac{z^4 + 2i}{z^3 + z} dz & \quad b) \quad \int_{\gamma} \frac{1}{e^z(z^2 + 1)^2} dz.
   \end{align*}

4. Given that $f$ is analytic in the punctured disc $\Omega = \{ z : 0 < |z - z_0| < R \}$ and $|f(z)| \leq M$ for some nonnegative constant $M$. Prove that $\int_{\gamma} f(z)dz = 0$ where $\gamma$ is any circle around $z_0$. What happens if $\gamma$ is any closed path in $\Omega$?

5. Let $\Omega$ be a k-connected Jordan domain and $z_0 \in \Omega$. Suppose that $f$ is an analytic function on a domain $\Omega'$ containing $\Omega$. Show that

   \[ f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{(z-z_0)^{n+1}} dz. \]

   Here $\partial \Omega$ is the union of simple Jordan curves comprising the boundary of $\Omega$. 

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