Nonhomogeneous ODEs-Undetermined Coefficient Method

Recall that the general solution of a nonhomogeneous ODE is given by

\[ y = y_c + y_p \]

where \( y_c \) is the **complimentary solution** (the general solution of the corresponding homogeneous ODE) and \( y_p \) is a **particular solution** (of the nonhomogeneous equation).

In this worksheet, we will use the **method of undetermined coefficients** to find \( y_p \).

> restart:

### Example 1

Consider the following 2nd order ODE

\[ ode := 2 \frac{d^2}{dt^2} y(t) - 3 \frac{d}{dt} y(t) - 5 y(t) = e^{2t} \]  \hspace{1cm} (1.1)

The corresponding homogeneous equation is (setting the right hand side to zero)

\[ hom_ode := 2 \frac{d^2}{dt^2} y(t) - 3 \frac{d}{dt} y(t) - 5 y(t) = 0 \]  \hspace{1cm} (1.2)

whose characteristic equation is

\[ odechar := 2X^2 - 3X - 5 = 0 \]  \hspace{1cm} (1.3)

The roots of this equation are the characteristic values

\[ \text{solve}(odechar, X); \quad \frac{5}{2}, -1 \]  \hspace{1cm} (1.4)

The roots are real and distinct. We then have the general solution of the homogeneous equation be given by

\[ y_c := t \rightarrow C[1] e^{-t} + C[2] e^{5/2t} \]  \hspace{1cm} (1.5)

2 is not a characteristic value so that we will seek for a particular solution \( y_p \) of the form (A is a constant needs to be determined)

\[ y_p := t \rightarrow A e^{2t} \]  \hspace{1cm} (1.6)

Substitute this into the original equation

\[ \text{subs}(y(t)=y_p(t), ode); \]  \hspace{1cm} (1.7)
\[ 2 \left( \frac{\partial^2}{\partial r^2} (A e^{2r}) \right) - 3 \left( \frac{\partial}{\partial r} (A e^{2r}) \right) - 5 A e^{2r} = e^{2r} \]  

(1.7)

Simplify the result

\[ > \text{simplify}(\%); \quad -3 A e^{2r} = e^{2r} \]  

(1.8)

From this, we solve for A

\[ > A := \text{solve}(\%, A); \quad A := -\frac{1}{3} \]  

(1.9)

\[ > y_p(t); \quad -\frac{1}{3} e^{2t} \]  

(1.10)

Thus the general solution is given by

\[ > y_g := t \rightarrow y_c(t) + y_p(t); \quad y_g(t) := t \rightarrow y_c(t) + y_p(t) \]  

(1.11)

\[ > y_g(t); \quad C_1 e^{-t} + C_2 e^{\frac{5}{2}t} - \frac{1}{3} e^{2t} \]  

(1.12)

\[ \textbf{Example 2} \]

Consider the following 2nd order ODE

\[ > \text{restart}; \]  

(2.1)

\[ > \text{ode} := \text{diff}(y(t), t$2) - 3*\text{diff}(y(t), t) = (t^2+t) \cdot \exp(t); \quad ode := \frac{d^2}{dt^2} y(t) - 3 \left( \frac{d}{dt} y(t) \right) = (t^2 + t) e^{t} \]  

The corresponding homogeneous equation is (setting the right hand side to zero)

\[ > \text{hom_ode} := \text{diff}(y(t), t$2) - 3*\text{diff}(y(t), t) = 0; \quad \text{hom}_ode := \frac{d^2}{dt^2} y(t) - 3 \left( \frac{d}{dt} y(t) \right) = 0 \]  

(2.2)

whose characteristic equation is

\[ > \text{odechar} := X^2 - 3X = 0; \quad odechar := X^2 - 3X = 0 \]  

(2.3)

The roots of this equation are the characteristic values

\[ > \text{solve}(\text{odechar}, X); \quad 0, 3 \]  

(2.4)

The roots are real and distinct. We then have the general solution of the homogeneous equation be given by

\[ > y_c := t \rightarrow C[1] + C[2] \cdot \exp(3t); \]  

(2.5)
\[ y_c := t \to C_1 + C_2 e^{3t} \]  

(2.5) 

1 is not a characteristic value so that we will seek for a particular solution \( y_p \) of the form (\( A, B, C \) are constants need to be determined) 
\[ y_p := t \to (A r^2 + B t + C) e^t \]  

(2.6) 

Substitute this into the original equation 
\[ \frac{\partial^2}{\partial t^2} ((A r^2 + B t + C) e^t) - 3 \left( \frac{\partial}{\partial t} ((A r^2 + B t + C) e^t) \right) = (r^2 + t) e^t \]  

(2.7) 

Simplify the result 
\[ 2A - 2At - B - 2Ar^2 - 2Bt - 2C = r^2 + t \]  

(2.8) 

Comparing the coefficients of powers of \( t \) on both sides we have the following system 
\[ \text{sys} := \{-2A = 1, -2B - 2A = 1, 2A - 2C - B = 0\}; \]  

(2.10) 

From this, we solve for \( A, B, C \) 
\[ \text{Const} := \text{solve(\text{sys}, \{A, B, C\})}; \]  

(2.11) 

Thus the general solution is given by 
\[ y_c(t) + Y_p; \]  

(2.13) 

\[ C_1 + C_2 e^{3t} + \left(-\frac{1}{2} r^2 - \frac{1}{2}\right) e^t \]  

Example 3 

Consider the following 2nd order ODE 
\[ \text{ode} := \frac{d^2}{dt^2} y(t) - 2y(t) = (2t+1) * \exp(-t); \]  

(3.1) 

The corresponding homogeneous equation is (setting the right hand side to zero) 
\[ \text{hom_ode} := \frac{d^2}{dt^2} y(t) - 2y(t) = 0; \]  

(3.2)
\[ \text{hom}_\text{ode} := \frac{d^2}{dt^2} y(t) - \left( \frac{d}{dt} y(t) \right) - 2 y(t) = 0 \]  

whose characteristic equation is

\[ \text{odechar} := X^2 - X - 2 = 0 \]  

The roots of this equation are the characteristic values

\[ \text{solve(odechar, X)}; \]

\[ 2, -1 \]

The roots are real and distinct. We then have the general solution of the homogeneous equation be given by

\[ y_c := t \rightarrow C[1] e^{-t} + C[2] e^{2t} \]  

-1 is a characteristic value (of multiplicity 1) so that we will seek for a particular solution \( y_p \) of the form (A, B are constants need to be determined). Notice the factor \( t \) in the front.

\[ y_p := t \rightarrow t (A t + B) e^{-t} \]  

Substitute this into the original equation

\[ \text{subs}(y(t)=y_p(t), \text{ode}); \]

\[ \frac{\partial^2}{\partial t^2} \left( t (A t + B) e^{-t} \right) - \left( \frac{\partial}{\partial t} \left( t (A t + B) e^{-t} \right) \right) - 2 t (A t + B) e^{-t} = (2 t + 1) e^{-t} \]  

Simplify the result

\[ \text{simplify(\%)/exp(-t)}; \]

\[ 2 A - 6 A t - 3 B = 2 t + 1 \]

\[ \text{collect(\%,t)}; \]

\[ 2 A - 6 A t - 3 B = 2 t + 1 \]

Comparing the coefficients of powers of \( t \) on both sides we have the following system

\[ \text{sys} := \{-6 A = 2, 2 A - 3 B = 1\}; \]

\[ \text{sys} := \{-6 A = 2, 2 A - 3 B = 1\} \]

From this, we solve for A

\[ \text{Const} := \text{solve(sys, \{A, B\})}; \]

\[ \text{Const} := \left\{ A = -\frac{1}{3}, B = -\frac{5}{9} \right\} \]

\[ \text{Y_p} := \text{subs(C, y_p(t))}; \]

\[ Y_p := t \left( -\frac{1}{3} t - \frac{5}{9} \right) e^{-t} \]

Thus the general solution is given by

\[ \text{y_g} := t \rightarrow y_c(t) + Y_p; \]

\[ y_g := t \rightarrow y_c(t) + Y_p \]
Example 4

Consider the following 2nd order ODE

\[\text{ode} := \frac{d^2}{dt^2} y(t) - 4 \left( \frac{d}{dt} y(t) \right) + 4 y(t) = (t^2 + t + 3) e^{2t}\]  

(4.1)

The corresponding homogeneous equation is (setting the right hand side to zero)

\[\text{hom_ode} := \frac{d^2}{dt^2} y(t) - 4 \left( \frac{d}{dt} y(t) \right) + 4 y(t) = 0\]  

(4.2)

whose characteristic equation is

\[\text{odechar} := X^2 - 4X + 4 = 0\]  

(4.3)

The roots of this equation are the characteristic values

\[\text{solve(odechar, X)};\]

\[2, 2\]  

(4.4)

The roots are real and double. We then have the general solution of the homogeneous equation be given by

\[y_c := t \rightarrow C[1] e^{2t} + C[2] t e^{2t}\]  

(4.5)

2 is a characteristic value (of multiplicity 2) so that we will seek for a particular solution \(y_p\) of the form \((A, B)\) are constants need to be determined). Notice the factor \(t^2\) in the front.

\[y_p := t \rightarrow t^2 (A t^2 + B t + C) e^{2t}\]  

(4.6)

Substitute this into the original equation

\[\frac{\partial^2}{\partial t^2} \left( t^2 (A t^2 + B t + C) e^{2t} \right) - 4 \left( \frac{\partial}{\partial t} \left( t^2 (A t^2 + B t + C) e^{2t} \right) \right) + 4 t^2 (A t^2 + B t + C) e^{2t} = (t^2 + t + 3) e^{2t}\]  

(4.7)

Simplify the result

\[\text{simplify(\%/exp(2*t))};\]

\[12 A t^2 + 6 B t + 2 C = t^2 + t + 3\]  

(4.8)
> collect(%1,t);
\[ 12 A t^2 + 6 B t + 2 C = t^2 + t + 3 \] (4.9)

Comparing the coefficients of powers of \( t \) on both sides we have the following system

> sys:={12*A=1, 6*B=1, 2*C=3};
\[ \text{sys} := \{12 A = 1, 6 B = 1, 2 C = 3\} \] (4.10)

From this, we solve for \( A \)

> Const:=solve(sys);
\[ \text{Const} := \left\{ C = \frac{3}{2}, B = \frac{1}{6}, A = \frac{1}{12} \right\} \] (4.11)

> Y_p:=subs(Const, y_p(t));
\[ Y_p := t^2 \left( \frac{1}{12} t^2 + \frac{1}{6} t + \frac{3}{2} \right) e^{2t} \] (4.12)

Thus the general solution is given by

> y_g:=t->y_c(t)+Y_p;
\[ y_g := t \rightarrow y_c(t) + Y_p \] (4.13)

> y_g(t);
\[ C_1 e^{2t} + C_2 t e^{2t} + t^2 \left( \frac{1}{12} t^2 + \frac{1}{6} t + \frac{3}{2} \right) e^{2t} \] (4.14)

Checking our solution

> simplify(subs(y(t)=y_g(t), ode));
\[ (t^2 + t + 3) e^{2t} = (t^2 + t + 3) e^{2t} \] (4.15)

\section*{Example 5}

Consider the following 2nd order ODE

> restart:
\[ \text{ode} := \frac{d^2}{dt^2} y(t) - 2 \left( \frac{d}{dt} y(t) \right) + 2 y(t) = t \cos(2 t) \] (5.1)

The corresponding homogeneous equation is (setting the right hand side to zero)

> hom_ode:=diff(y(t), t$2) - 2*diff(y(t),t)+2*y(t)=0;
\[ \text{hom_ode} := \frac{d^2}{dt^2} y(t) - 2 \left( \frac{d}{dt} y(t) \right) + 2 y(t) = 0 \] (5.2)

whose characteristic equation is

> odechar:=X^2-2*X+2=0;
\[ \text{odechar} := X^2 - 2 X + 2 = 0 \] (5.3)

The roots of this equation are the characteristic values

> solve(odechar, X);
\[ 1 + i, 1 - i \] (5.4)
The roots are conjugate complex. We then have the general solution of the homogeneous equation be given by

\[ y_c(t) = a_1 e^{t} \cos(t) + a_2 e^{t} \sin(t) \tag{5.5} \]

From the right hand side we see that \( \cos(2t) \) comes from \( e^{(0+2i)t} \) and 0 + 2 \( i \) is NOT a characteristic value so that we will seek for a particular solution \( y_p \) of the form (A,B,C,E are constants need to be determined). Notice the presence of \( \sin(2t) \) in the formula. We have to avoid the use of \( D \) since it is used for differential operator in Maple.

\[ y_p(t) = (A t + B) \cos(2t) + (C t + E) \sin(2t) \tag{5.6} \]

Substitute this into the original equation

\[ \frac{d^2}{dt^2} \left( (A t + B) \cos(2t) + (C t + E) \sin(2t) \right) - 2 \left( \frac{d}{dt} \left( (A t + B) \cos(2t) + (C t + E) \sin(2t) \right) \right) + 2 (A t + B) \cos(2t) + 2 (C t + E) \sin(2t) = t \cos(2t) \tag{5.7} \]

Simplify the result

\[ \text{simplify}(%); \]

\[ -4 A \sin(2t) - 2 \cos(2t) A t - 2 \cos(2t) B + 4 C \cos(2t) - 2 \sin(2t) C t - 2 \sin(2t) E A t - 2 \cos(2t) + 4 \sin(2t) A t + 4 \sin(2t) B - 2 C \sin(2t) - 4 \cos(2t) C t - 4 \cos(2t) E = t \cos(2t) \tag{5.8} \]

Comparing the coefficients of powers of \( \cos(2t) \), \( \sin(2t) \) and \( t \) on both sides we have the following system

\[ \text{sys:} = \{-2 A - 4 C = 1, -2 B - 2 A + 4 C - 4 E = 0, -2 C + 4 A = 0, -4 A - 2 E + 4 B = 0 \}; \]

\[ \text{sys:} = \{ -2 A - 4 C = 1, -4 E - 2 B + 4 C - 2 A = 0, -2 C + 4 A = 0, -4 A - 2 E + 4 B = 0 \} \tag{5.10} \]

From this, we solve for A

\[ \text{Const:=solve(sys)}; \]

\[ \text{Const := \{B = -\frac{11}{50}, E = -\frac{1}{25}, A = -\frac{1}{10}, C = -\frac{1}{5} \}} \tag{5.11} \]

\[ \text{Y_p:=subs(Const, y_p(t))}; \]

\[ Y_p := \left( -\frac{1}{10} t - \frac{11}{50} \right) \cos(2t) + \left( -\frac{1}{5} t - \frac{1}{25} \right) \sin(2t) \tag{5.12} \]

Thus the general solution is given by

\[ y_g(t) = y_c(t) + Y_p; \tag{5.13} \]
\[ y_g := t \to y_c(t) + Y_p \] (5.13)

\[ y_g(t) := a_1 e^t \cos(t) + a_2 e^t \sin(t) + \left( -\frac{1}{10} t - \frac{11}{50} \right) \cos(2t) + \left( -\frac{1}{5} t - \frac{1}{25} \right) \sin(2t) \] (5.14)