**Sturm-Liouville Eigenvalue Problems and Generalized Fourier Series**

**Examples of Regular Sturm-Liouville Eigenvalue Problems**

We will now look at examples of regular **Sturm-Liouville differential equations** with various combinations of the three types of **boundary conditions** Dirichlet, Neumann and Robin. All of the examples are special cases of the Sturm-Liouville differential equation

\[ L(y) + \lambda w(x) y = 0 \]

where \( L \) is the **Sturm-Liouville operator**

\[ L(y) = \left( \frac{\partial}{\partial x} p(x) \left( \frac{\partial}{\partial x} y \right) \right) + q(x) y \]

We focus on three types of differential equations: Euler, Cauchy-Euler, and Bessel. Each one of these differential equations is characterized by a different set of Sturm-Liouville coefficients \( p(x), q(x), \) and \( w(x) \). Subject to a particular set of boundary conditions, we will:

1. generate the **eigenvalues**, the corresponding **eigenfunctions**, and the statement of **orthonormality**.
2. then provide an example of a **generalized Fourier series** expansion of a given function in terms of the particular eigenfunctions.

In solving for the allowed eigenvalues and corresponding eigenfunctions, we would ordinarily consider three possibilities for values of \( \lambda \): \( \lambda < 0 \), \( \lambda = 0 \), and \( \lambda > 0 \). However, to make our task a little simpler, we will not consider the case for \( \lambda < 0 \) because it can be shown, by way of the **Rayleigh quotient**, that, for the particular Sturm-Liouville problems we will be considering, \( \lambda \) must be greater than or equal to zero.

In this worksheet we will look at the **Euler operator** \( Lu = \frac{\partial^2}{\partial x \partial x} u \) on the interval \((0,b)\) and the BC are

1/ Dirichlet condition at both ends.
2/ Mixed BC: Dirichlet at 0 and Neumann at b.

**EXAMPLE 2.5.1:** Consider the **Euler operator with Dirichlet conditions**. We seek the eigenvalues and corresponding orthonormal eigenfunctions for the Euler differential equation [Sturm-Liouville type for \( p(x) = 1 \), \( q(x) = 0 \), \( w(x) = 1 \)] over the interval \( I = \{ x \mid 0 < x < b \} \). The boundary conditions are type 1 at the left and type 1 at the right end points.
Euler differential equation

\[ \frac{d^2}{dx \, dx} y(x) + \lambda \, y(x) = 0 \]

Boundary conditions

\[ y(0) = 0 \quad \text{and} \quad y(b) = 0 \]

\[ \text{SOLUTION: We consider two possibilities for values of } \lambda. \]

\[ \text{We first consider } \lambda = 0. \]

The system basis vectors (or fundamental solutions) are

\[ \text{restart: y1 := } x \rightarrow 1; \text{ y2 := x } \rightarrow x; \]

\[ y1 := 1 \]
\[ y2 := x \rightarrow x \]

General solution is then

\[ y := x \rightarrow C1 \, y1(x) + C2 \, y2(x); \]

\[ y := x \rightarrow C1 \, y1(x) + C2 \, y2(x) \]

Substitution into the boundary conditions yields the system to determine C1, C2

\[ \{y(0) = 0, \ y(b) = 0\}; \text{ solve(%, \{C1, C2\});} \]

\[ \{ C1 = 0, \ C1 + C2 \, b = 0 \} \]
\[ \{ C2 = 0, \ C1 = 0 \} \]

We obtain the trivial solution \( y(x) = 0 \), thus \( \lambda = 0 \) is not an eigenvalue.

\[ \text{We next consider } \lambda > 0. \text{ We set } \lambda = \mu^2. \]

The system basis vectors are

\[ \text{y1 := x } \rightarrow \sin(\mu \, x); \text{ y2 := x } \rightarrow \cos(\mu \, x); \]

\[ y1 := x \rightarrow \sin(\mu \, x) \]
\[ y2 := x \rightarrow \cos(\mu \, x) \]

General solution

\[ y := x \rightarrow C1 \, y1(x) + C2 \, y2(x); \]

\[ y := x \rightarrow C1 \, y1(x) + C2 \, y2(x) \]

Substituting into the boundary conditions, we get

\[ \{y(0) = 0, \ y(b) = 0\}; \]
From the first equation, $C_2 = 0$. We must look for $\mu$ such that $y(x)$ is not identically zero. The only nontrivial solutions to the above occur when $C_2 = 0$, $C_1$ is arbitrary and $\mu$ satisfies the following eigenvalue equation.

Let's ask Maple to find $\mu$:

```maple
> sin(mu*b)=0; solve(%,mu);
sin(\mu b) = 0
0
```

Maple did not give us all of possible solutions. In fact, $\sin(x) = 0$ if $x = n\pi$ for integers $n\mu$. Thus, $\mu$ takes on the values

```maple
> mu:=n->n*Pi/b;
\mu := n \rightarrow \frac{n \pi}{b}

for $n = 1, 2, 3, \ldots$.
```

Allowed eigenvalues are $\lambda_n = \mu_n^2$

```maple
> lambda[n]:=(n*Pi/b)^2;
\lambda_n := \frac{n^2\pi^2}{b^2}
```

Non-normalized eigenfunctions are

```maple
> Phi:=(n,x)->sin(mu(n)*x);
\Phi := (n, x) \rightarrow \sin(\mu(n) x)
```

Normalization

Evaluating the norm from the inner product of the eigenfunctions with respect to the weight function $w(x) = 1$ over the interval yields

```maple
> w(x):=1:sqrt(int(Phi(n,x)^2*w(x),x=0..b));
1
2
b
2
\sqrt{-2 \cos(n \pi) \sin(n \pi) + n \pi}
2
\mu

Substitution of the eigenvalue equation simplifies the norm

```maple
> e_norm[n]:=radsimp(subs({sin(n*Pi)=0,cos(n*Pi)=(-1)^n},%));
```
Orthonormal eigenfunctions is then obtained by dividing the non-normalized eigenfunctions by their norm

\[ \phi := (n, x) \rightarrow \frac{\Phi(n, x)}{e_{\text{norm}}[n]} \]

Statement of orthonormality

\[ \int_0^b \sin \left( \frac{n \pi x}{b} \right) \sqrt{2} \sin \left( \frac{m \pi x}{b} \right) \frac{1}{\sqrt{b e_{\text{norm}}[m]}} \, dx = \delta(n, m) \]

Fourier coefficients

\[ F_n := \int_0^b \frac{f(x) \sin \left( \frac{n \pi x}{b} \right) \sqrt{2}}{\sqrt{b}} \, dx \]

Generalized Fourier series expansion

\[ f_{\text{series}} := x \rightarrow \sum_{n=1}^{\infty} F_n \phi(n, x) \]

This is the generalized series expansion of \( f(x) \) in terms of the "complete" set of eigenfunctions for the particular Sturm-Liouville operator and given boundary conditions over the interval.

DEMONSTRATION: Develop the generalized series expansion for \( \tilde{f}(x) = x \) over the interval \( I = \{ x \mid \]

\[ e_{\text{norm}} := \frac{1}{2} \sqrt{2 \sqrt{b}} \]
0 < x < 1 \} in terms of the above eigenfunctions. We assign the system values

```plaintext
> a:=0;b:=1;f:=x->x;

a := 0
b := 1
f := x \rightarrow x
```

**SOLUTION:** We evaluate the Fourier coefficients

```plaintext
> eval(int(f(x)*phi(n,x)*w(x),x=a..b));

\[-2 \left( \frac{-\sin(n \pi) + n \pi \cos(n \pi)}{n^2 \pi^2} \right)\]

> F[n] := subs({\sin(n*Pi)=0, \cos(n*Pi)=(-1)^n},%);

\[F_n := -\frac{\sqrt{2} (-1)^n}{n \pi}\]

> Series := x -> sum(F[n]*phi(n,x),n=1..infinity);

\[Series := x \rightarrow \sum_{n=1}^{\infty} F_n \phi(n,x)\]

First five terms of expansion

```plaintext
> Part_Series := (m,x) -> sum(F[n]*phi(n,x),n=1..m);

\[Part_Series := (m,x) \rightarrow \sum_{n=1}^{m} F_n \phi(n,x)\]

> Part_Series(2,.5);

.6366197722

> plot({Part_Series(5,x), f(x)},x=0..1, thickness=3);
```
The curves of Figure 2.4 depict the function $f(x)$ and its Fourier series approximation in terms of the orthonormal eigenfunctions for the particular operator and boundary conditions given earlier. Note that $f(x)$ satisfies the given boundary conditions at the left but fails to do so at the right end point. The convergence is pointwise.

**EXAMPLE 2.5.2:** Consider the Euler operator with **Dirichlet and Neumann conditions**. We seek the eigenvalues and corresponding orthonormal eigenfunctions for the Euler differential equation [Sturm-Liouville type for $p(x) = 1$, $q(x) = 0$, $w(x) = 1$] over the interval $I = \{ x \mid 0 < x < b \}$. The boundary conditions are type 1 at the left and type 2 at the right.

Euler differential equation

$$\left( \frac{\partial^2}{\partial x \partial x} y(x) \right) + \lambda y(x) = 0$$

Boundary conditions
SOLUTION: We consider two possibilities for values of $\lambda$. We first consider $\lambda = 0$.

The system basis vectors are

> restart: y1 := x -> 1; y2 := x -> x;

General solution

> y := x -> C1*y1(x) + C2*y2(x);

Substituting the boundary conditions yields

> subs(x=0, y(x)) = 0;

> subs(x=b, diff(y(x), x)) = 0;

The only solution to the above is the trivial solution. We next consider $\lambda > 0$. We set $\lambda = \mu^2$.

The system basis vectors are

> y1 := x -> sin(mu*x); y2 := x -> cos(mu*x);

General solution

Substituting the boundary conditions yields

> eval(subs(x=0, y(x))) = 0;

> eval(subs(x=b, diff(y(x), x))) = 0;
\[ C1 \cos(\mu b) \mu - C2 \sin(\mu b) \mu = 0 \]

The only nontrivial solutions occur when \( C2 = 0 \), \( C1 \) is arbitrary, and \( \mu \) satisfies the following eigenvalue equation

\[ \cos(\mu b) = 0 \]

Thus, \( \mu \) takes on values

\[ \mu_n := \frac{(2n-1)\pi}{2b} \]

for \( n = 1, 2, 3, \ldots \)

Allowed eigenvalues are \( \lambda_n = \mu_n^2 \)

\[ \lambda_n := \frac{(2n-1)^2\pi^2}{b^2} \]

Non-normalized eigenfunctions are

\[ \Phi := (n, x) \rightarrow \sin(\mu_n x) \]

Normalization

Evaluating the norm from the inner product of the eigenfunctions with respect to the weight function

\[ w(x) := 1 \]

yields

\[ e_{\text{norm}} := \frac{1}{2} \sqrt{2} \sqrt{\frac{2 (\sin(\pi n) \cos(\pi n) + 2 \pi n - \pi)}{(2n-1)\pi}} \]

Substitution of the eigenvalue equation simplifies the norm

\[ e_{\text{norm}} := \frac{1}{2} \sqrt{2} \sqrt{b} \]
Orthonormal eigenfunctions

\[ \phi := (n,x) \rightarrow \frac{\Phi(n,x)}{e_{\text{norm}}[n]} \]

Statement of orthonormality

\[ \text{Int} \left( \phi(n,x) \ast \phi(m,x) \ast w(x), x=0..b \right) = \delta(n,m); \]

\[ \int_{0}^{b} \frac{\sin \left( \frac{1}{2} \frac{(2n-1)\pi x}{b} \right)}{\sqrt{2}} \frac{\sin \left( \mu_m x \right)}{\sqrt{b} e_{\text{norm}}[m]} \, dx = \delta(n,m) \]

Fourier coefficients

\[ F[n] := \text{int} (f(x) \ast \phi(n,x) \ast w(x), x=0..b); \]

\[ F_n := \int_{0}^{b} \frac{f(x) \sin \left( \frac{1}{2} \frac{(2n-1)\pi x}{b} \right)}{\sqrt{2}} \frac{1}{\sqrt{b}} \, dx \]

Generalized Fourier series expansion

\[ f_{\text{Series}} := x \rightarrow \sum_{n=1}^{\infty} F_n \phi(n,x); \]

This is the generalized series expansion of \( f(x) \) in terms of the "complete" set of eigenfunctions for the particular Sturm-Liouville operator and boundary conditions over the interval.

\[ f_{\text{Series}} := x \rightarrow \sum_{n=1}^{\infty} F_n \phi(n,x) \]

**DEMONSTRATION**: Develop the generalized series expansion for \( f(x) = x \) over the interval \( I = \{ x \mid 0 < x < 1 \} \) in terms of the preceding eigenfunctions. We assign the system values
a := 0; b := 1; f := x → x;

\[ F_n := -2 \frac{\sqrt{2} (2 \cos(\pi n) + 2 \pi \sin(\pi n) n - \pi \sin(\pi n))}{(2 n - 1)^2 \pi^2} \]

SOLUTION: We evaluate the Fourier coefficients

\[ F[n] := \text{int}(f(x) \cdot \phi(n, x) \cdot w(x), x=a..b); \]

\[ F_n := -4 \frac{\sqrt{2} (-1)^n}{(2 n - 1)^2 \pi^2} \]

\[ f_Series := \text{Sum}(F[n] \cdot \phi(n, x), n=1..\infty); \]

\[ f_Series := \sum_{n=1}^{\infty} \left( (-1)^n \frac{\sin \left( \frac{1}{2} (2 n - 1) \pi x \right)}{(2 n - 1)^2 \pi^2} \right) \]

Partial expansion

\[ Part_Series := (m, x) \rightarrow \text{sum}(F[n] \cdot \phi(n, x), n=1..m); \]

\[ Part_Series := (m, x) \rightarrow \sum_{n=1}^{m} F_n \phi(n, x) \]

\[ \text{plot}\{\{Part_Series(3, x), f(x)\}, x=0..b, \text{thickness}=3\}; \]
The two curves of Figure 2.5 depict the function $f(x)$ and its Fourier series approximation in terms of the orthonormal eigenfunctions for the particular operator and boundary conditions given earlier. Note that $f(x)$ satisfies the given boundary conditions at the left but fails to do so at the right end point. The convergence is pointwise.