We now prove some convergence theorems for Lebesgue’s integral. The main question is this. Let \( \{f_n\} \) be a sequence of Lebesgue integrable functions on \( E \) and assume that \( f_n \) converges a.e. to \( f \) on \( E \). Is \( f \) Lebesgue integrable and
\[
\lim \int_E f_n = \int_E f ?
\]

Let’s look at the following examples. Define the following functions on \([0, 1]\).
\[
f_n(x) = \frac{1}{x} \chi_{[1/n, 1]}(x) \text{ and } g_n(x) = n \chi_{[0, 1/n]}(x).
\]

We easily see that \( f_n \rightarrow f \) with \( f(x) = 1/x \) if \( x > 0 \) and \( f(0) = 0 \). But \( f \) is not Lebesgue integrable (why?)! Meanwhile, \( g_n \rightarrow g \) with \( g \equiv 0 \). But
\[
\int_{[0, 1]} g = 0 \neq 1 = \lim \int_{[0, 1]} g_n.
\]

We see that appropriate hypotheses need to be assumed to answer our question.

1 Convergence theorems

We start with the monotone convergence theorem.

**Theorem 1.1** If \( f_n \) is a sequence of nonnegative measurable functions defined on \( E \) and \( f_n \uparrow f \) on \( E \) then \( \lim \int_E f_n = \int_E f \).

**Proof:** Obviously, \( \int_E f \geq \lim \int_E f_n \) (why?). To prove the opposite inequality, for each integer \( m \) choose an increasing sequence \( \{s_{m,n}\} \) of simple functions so that \( s_{m,n} \uparrow f_m \) as \( n \rightarrow \infty \). We then define
\[
S_n(x) = \sup\{s_{k,n}(x) : 1 \leq k \leq n\}, \text{ for } n \geq 1.
\]

Because \( s_{k,n} \leq s_{k,n+1} \), we have (check it!)
\[
S_n \leq S_{n+1} \text{ and } s_{m,n} \leq S_n \leq f_n \text{ for } 1 \leq m \leq n.
\]

Letting \( n \rightarrow \infty \) in the last inequality, we get
\[
f_m \leq \lim_{n \rightarrow \infty} S_n \leq f
\]
for each \( m \in N \). Let \( m \rightarrow \infty \) in the above inequality, we see that \( S_n \uparrow f \). Therefore
\[
\lim \int_E S_n = \int_E f.
\]
On the other hand, since \( s_{k,n} \leq f_k \leq f_n \) for all \( k \leq n \), we have \( S_n \leq f_n \). Hence

\[
\int_E f = \lim \int_E S_n \leq \lim \int_E f_n.
\]

This completes the proof. \( \blacksquare \)

Next, we present the **Fatou lemma**, which may not seem very interesting but its applications in different contexts will be seen apparent later.

**Lemma 1.2** (Fatou's lemma) If \( f_n \) is a sequence of nonnegative measurable functions defined on \( E \), then

\[
\liminf_{n \to \infty} \int_E f_n \geq \int_E \liminf_{n \to \infty} f_n.
\]

**Proof:** The function \( f = \liminf_{n \to \infty} f_n \) is nonnegative and measurable (why?). Set \( h_m = \inf_{n \geq m} f_n \). Then \( f_m \geq h_m \) and \( h_m \uparrow f \). By the above theorem, we have

\[
\int_E f = \lim \int_E h_m.
\]

But

\[
\int_E h_m \leq \inf_{n \geq m} \int_E f_n \implies \lim \int_E h_m \leq \liminf \int_E f_n.
\]

The proof is complete. \( \blacksquare \)

We also have (prove this!)

**Lemma 1.3** If \( f_n \) is a sequence of measurable functions defined on \( E \) and there is a Lebesgue integrable function \( u \) such that \( f_n \leq u \) for all \( n \), then

\[
\limsup_{n \to \infty} \int_E f_n \leq \int_E \limsup_{n \to \infty} f_n.
\]

The third convergence theorem is the **Lebesgue dominated convergence theorem**.

**Theorem 1.4** Let \( \{f_n\} \) be a sequence of measurable functions on \( E \) that converges to \( f \) a.e. on \( E \). Suppose that there exists a Lebesgue integrable function \( g \) on \( E \) such that \( |f_n| \leq g \) for all \( n \). Then \( f \) is Lebesgue integrable and

\[
\lim \int_E |f_n - f| = 0.
\]

In particular, we have \( \lim \int_E f_n = \int_E f \).

**Proof:** Since \( |f_n| \leq |g| \) and \( f_n \to f \) we see that \( |f| \leq g \). Thus, \( |f_n - f| \leq |f_n| + |f| \leq 2g \). Because \( |f_n - f| \to 0 \) and \( 2g \) is Lebesgue integrable, Lemma 1.3 yields

\[
\limsup \int_E f - \int_E f_n \leq \limsup \int_E |f_n - f| \leq \int_E \limsup |f_n - f| = 0.
\]

This gives the theorem. \( \blacksquare \)
2 Other theorems

We present here another version of the fundamental theorem we learn in calculus. Much more general versions of this are available but they are out of the scope of this course.

Let’s start with the following

Lemma 2.1 Let $f : [a, b] \rightarrow \mathbb{R}$ be a measurable function. Assume that $f$ is differentiable a.e. on $[a, b]$. Then $f'$ is measurable.

Proof: Extend the function $f$ to the interval $[a, b+1]$ by setting $f$ to be a constant on $[b, b+1]$. For each integer $n$, define the function $f_n : [a, b] \rightarrow \mathbb{R}$ by

$$f_n(x) = n(f(x + \frac{1}{n}) - f(x)).$$

Obviously, $f_n$ is measurable. Moreover, $f_n(x)$ converges to $f'(x)$ (why?) whenever $f'(x)$ is defined. Hence, $f_n$ converges to $f'$ a.e. and, therefore, $f'$ is measurable.

The proof of the following simple lemma is left as a homework.

Lemma 2.2 Let $f : [a, b] \rightarrow \mathbb{R}$ be a measurable function. If $f$ is continuous at $x \in [a, b)$ then

$$\lim_{n \rightarrow \infty} n \int_x^{x+\frac{1}{n}} f = f(x).$$

Lemma 2.3 Let $a \in \mathbb{R}$ and $E \subset \mathbb{R}$. Let $f : E \cup (E + a) \rightarrow \mathbb{R}$ be a measurable function. We have

$$\int_E f(x + a) = \int_{E+a} f.$$

Proof: Using the translation invariant property of Lebesgue measure to prove this for simple functions. The general case follows by limit argument (your homework!).

We then have the following version of the fundamental theorem of calculus.

Theorem 2.4 Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function. If $f'$ is bounded, then $f'$ is Lebesgue integrable on $[a, b]$. Moreover, for any $c \in [a, b]$, we have

$$\int_a^c f'(x)dx = f(c) - f(a).$$

Proof: Let $M = \sup_{(a,b)} |f'(x)|$. We assume first that $c < b$. For $n$ sufficiently large, we consider the function $f_n$ defined on $[a, c]$ by

$$f_n(x) = n(f(x + \frac{1}{n}) - f(x)).$$

By the Mean Value Theorem, for each such integer $n$ and each $x \in [a, c]$, there exists $z_n^x \in (x, x + 1/n)$ such that

$$f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = f'(z_n^x) \leq M.$$
Thus, $f_n \leq M$, for all $n$. Since $f_n \to f'$ on $[a, c]$, Lebesgue dominated convergence theorem gives

$$\int_a^c f' = \lim_{n \to \infty} \int_a^c f_n.$$  

Using the above lemmas and the continuity of $f$ at $a, c$, we have

$$\int_a^c f' = \lim_{n \to \infty} \int_a^c f_n = \lim_{n \to \infty} n \left( \int_a^{c+1/n} f(x) - \int_a^c f(x) \right) = \lim_{n \to \infty} n \left( \int_c^{c+1/n} f(x) - \int_a^{a+1/n} f(x) \right) = \lim_{n \to \infty} n \int_c^{c+1/n} f(x) - \lim_{n \to \infty} n \int_a^{a+1/n} f(x) = f(c) - f(a).$$

This completes the proof for $c < b$. For $c = b$, we can extend the function $f$ to $a, b + 1]$ by defining $f(x) = f'(b)(x - b) + f(b)$ for $x \in (b, b + 1]$ and reduce this case to the one discussed before.

**Exercises**

1. Show that $f$ is Lebesgue integrable iff $f$ is Lebesgue integrable.

2. Let $f(x) = 1/\sqrt{x}$ for $x > 0$ and $f(0) = 0$. Show that $f$ is Lebesgue integrable.

3. Is $f(x) = 1/x$ Lebesgue integrable on $(0, 1)$?

4. Let $f : E \to \mathbb{R}$ be a measurable function. Assume that $f \neq 0$ a.e. on $E$ and that $\int_E f = 0$. Show that $f = 0$ a.e. on $E$.

5. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function. Compute $\lim_{n \to \infty} \int_0^1 x^n f(x)$.

6. For $a, b \in \mathbb{R}$, compute $\lim_{n \to \infty} \int_a^b \sin^n(x)dx$ and $\lim_{n \to \infty} \int_a^b \cos^n(x)dx$.

7. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $E$. Show that

$$\int_E \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int_E f_n.$$  

8. For each integer $n$ and $x > 0$, define $f_n(x) = \sqrt[n]{x}/x$. Show that

a) $f_n \to f$ pointwise with $f(x) = 1/x$ on the set $x > 0$.

b) $\{f_n\}$ is monotone on $(0, 1)$ and $(1, \infty)$.

c) For any $z > 0$, $\ln(z) = \lim_{n \to \infty} n(\sqrt[n]{z} - 1)$.

9. Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable. Prove that the function $F(x) = \int_a^x f$ is uniformly continuous on $[a, b]$.