Lebesgue Measure

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1 Introduction

How do we measure the "size" of a set in $\mathbb{R}$? Let's start with the simplest ones: intervals. Obviously, the natural candidate for a measure of an interval is its length, which is used frequently in differentiation and integration. For any bounded interval I (open, closed, half-open) with endpoints a and b ($a \le b$), the length of I is defined by $\ell(I) = b - a$. Of course, the length of any unbounded interval is defined to be infinite, that is, $\ell(I) = \infty$ if I is of the form (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$.

How do we measure the size of sets other than intervals? An extension to unions of intervals is obvious, but is much less obvious for arbitrary sets. For instance, what is the size of the set of irrational numbers in $[0, 2]$? Is it possible to extend this concept of length (or size) of an interval to arbitrary sets? **Lebesgue measure** is one of several approaches to solving this problem.

Here, we want to define and study the basic properties of Lebesgue measure. Given a set E of real numbers, $\mu(E)$ will represent its Lebesgue measure. Before defining this concept, let's consider the properties that it should have.

(1) If I is an interval, then $\mu(I)$ should naturally be $\ell(I)$, as we expected.

(2) If $A \subseteq B$, then $\mu(A) \le \mu(B)$. Smaller set, smaller measure.

(3) Given $A \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}$, define $A + x_0 = \{x + x_0 : x \in A\}$, the translation of A. Then $\mu(A + x_0) = \mu(A)$ since translation should not change the measure.

(4) If A and B are disjoint sets, then $\mu(A \bigcup B)$ should be the sum $\mu(A) + \mu(B)$. In fact, if $\{A_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} A_i)$ should be $\sum_{i=1}^{\infty} \mu(A_i)$.

Unfortunately, it is *not* possible to define a measure that satisfies all of these properties *for all subsets of real numbers.* The difficulty lies in property (4). Since this property is essential to guarantee the linearity of the Lebesgue integral (as we will see later), it is necessary to restrict the collection of sets and consider only those for which all of the properties are valid. In other words, some sets will *not* have a Lebesgue measure.

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2 Some facts on the topology of $\mathbb{R}$

Definition 2.1 *Let A be a subset of $\mathbb{R}$.*

1) *A is* **open** *in $\mathbb{R}$ if for each $x \in A$ there exists a neighborhood $V_\delta(x)$ such that $V_\delta(x) \subseteq A$.*

2) *A is* **closed** *in $\mathbb{R}$ if its complement $\mathbb{R} \backslash A$ is open in $\mathbb{R}$.*

The following basic results describe the manner in which open and closed sets relate to the operations of the union and intersection of sets.

Theorem 2.2 **i)** *The union of an arbitrary collection of open sets is open.*

ii) *The intersection of any finite collection of open sets is open.*

iii) *The intersection of an arbitrary collection of closed sets is closed.*

iv) *The union of any finite collection of closed sets is closed.*

Theorem 2.3 *A set A is closed iff any convergent sequence (x_n) of elements in A must have its limit $\lim x_n \in A$.*

An important class of subsets of $\mathbb{R}$ is the class of *compact sets*. We have the following definition.

Definition 2.4 *Let A be a subset of $\mathbb{R}$. An* open cover *of A is a collection $\{G_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains A; that is, $A \subseteq \bigcup_\alpha G_\alpha$.*

Definition 2.5 *A subset K of $\mathbb{R}$ is said to be compact iff* every *open cover of K has a* finite subcover.

Theorem 2.6 *(Heine-Borel theorem) A subset K of $\mathbb{R}$ is compact iff it is closed and bounded.*

Theorem 2.7 *A subset K of $\mathbb{R}$ is compact iff every sequence in K has a subsequence that converges to a point in K.*

The proof of the above facts can be found in Chapter 11 of the text.

Exercises

1. Show that every open set in $\mathbb{R}$ can be expressed as a countable union of open intervals (Hint: The set of rationals is countable).

3 Outer measure

Let's begin with a way to "measure" all sets.

Definition 3.1 *Let E be a subset of $\mathbb{R}$. The* Lebesgue <u>outer</u> measure *of E, denoted by $\mu^*(E)$, is defined by*

$$\inf\{\sum_k \ell(I_k) : \{I_k\} \text{ is a sequence of open intervals such that } E \subseteq \bigcup_k I_k\}.$$

If $\{I_k\}$ is a sequence of open intervals such that $E \subseteq \bigcup_k I_k$, we call $\{I_k\}$ a *cover* of E by open intervals. It is obvious that $0 \leq \mu^*(E) \leq \infty$.

Firstly, μ^* is almost a desired measure as it has the following properties.

Theorem 3.2 *Lebesgue outer measure has the following properties:*

(a) *If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.*

(b) *$\mu^*(\emptyset)$ is zero.*

(c) *If A is a countable set then $\mu^*(A)$ is zero.*

(d) *Lebesgue outer measure is invariant under translation, that is, for each real number x_0, $\mu^*(E + x_0) = \mu^*(E)$.*

(e) *Lebesgue outer measure is countably subadditive, that is, given a sequence of sets $\{E_i\}$, $\mu^*(\bigcup_i E_i) \leq \sum_i \mu^*(E_i)$.*

(f) *For any interval I, $\mu^*(I) = \ell(I)$.*

Proof: (a) and (b) are trivial (exercise).

Proof of (c): Let $A = \{x_i : i = 1, 2, \ldots\}$ be a countable set. Let $\varepsilon > 0$ and $\varepsilon_k = \varepsilon/2^{k+1}$, $k = 1, 2, \ldots$. Define $I_k = (x_k - \varepsilon_k, x_k + \varepsilon_k)$ then $A \subseteq \bigcup_k I_k$. Since

$$\sum_k \ell(I_k) = \sum_k 2\varepsilon_k \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \varepsilon.$$

It follows that $\mu^*(A) \leq \varepsilon$ for all $\varepsilon > 0$. Thus, $\mu^*(A) = 0$.

Proof of (d): Since each cover of the set E by open intervals I_k generates a cover of $E + x_0$ by open intervals $I_k + x_0$ with the same length. Thus, $\mu^*(E + x_0) \leq \mu^*(E)$. Similarly, $\mu^*(E) \leq \mu^*(E + x_0)$ since $E = (E + x_0) + (-x_0)$ is a translation of $E + x_0$. Thus, $\mu^*(E) = \mu^*(E + x_0)$.

Proof of (e): If $\sum_i \mu^*(E_i) = \infty$, then (e) is trivial. Suppose that the sum is finite. Let $\varepsilon > 0$ be given. For each i, there is a sequence $\{I_{ik}\}$ of open intervals such that $E_i \subseteq \bigcup_k I_{ik}$ and $\sum_k \ell(I_{ik}) < \mu^*(E_i) + \varepsilon/2^i$. Clearly, the doubly-indexed sequence $\{I_{ik}\}$ is a cover for $\bigcup_i E_i$. That is $\bigcup_i E_i \subseteq \bigcup_i \bigcup_k I_{ik}$. We have

$$\mu^*(\bigcup_i E_i) \leq \sum_{i,k} \ell(I_{ik}) = \sum_i \sum_k \ell(I_{ik}) \leq \sum_i (\mu^*(E_i) + \varepsilon/2^i)) = \sum_i \mu^*(E_i) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the above gives $\mu^*(\bigcup_i E_i) \le \sum_i \mu^*(E_i)$.

Proof of (f): Suppose first that I is bounded and its endpoints are a, b. For any given $\varepsilon > 0$, we have

$$I \subseteq (a,b) \bigcup (a-\varepsilon, a+\varepsilon) \bigcup (b-\varepsilon, b+\varepsilon).$$

Thus,

$$\mu^*(I) \le \ell((a,b)) + \ell((a-\varepsilon, a+\varepsilon)) + \ell((b-\varepsilon, b+\varepsilon)) = (b-a) + 2\varepsilon + 2\varepsilon = b - a + 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\mu^*(I) \le b - a = \ell(I)$.

We need only to prove that $\mu^*(I) \ge \ell(I)$. Firstly, consider the case when $I = [a,b]$. Let $\{I_k\}$ be any sequence of open intervals that covers I. Since I is compact, by the Heine-Borel theorem, there is a finite subcollection $\{J_i : 1 \le i \le n\}$ of $\{I_k\}$ still covers I. By reordering and deleting if necessary, we can assume that $a \in J_1 = (a_1, b_1)$,

$$b_1 \in J_2 = (a_2, b_2),\ b_2 \in J_3 = (a_3, b_3),\ \ldots,\ b_{n-1} \in J_n = (a_n, b_n),$$

and $b \in J_n$. We then have

$$b - a < b_n - a_1 = \sum_{i=2}^{n} (b_i - b_{i-1}) + (b_1 - a_1) < \sum_{i=1}^{n} \ell(J_i) \le \sum_i \ell(I_i).$$

Hence, $\ell(I) \le \mu^*(I)$. This proves the result when $I = [a,b]$.

If $I = (a,b)$ then, as we have shown above,

$$\ell([a,b]) = \mu^*([a,b]) \le \mu^*((a,b)) + \mu^*(a) + \mu^*(b) = \mu^*((a,b)).$$

Hence $\ell(I) = \ell([a,b]) \le \mu^*((a,b))$ as we desired. The case of half open intervals is treated similarly. The case of infinite intervals is left as exercise. ∎

From Definition 3.1, every set has a Lebesgue outer measure. The above theorem shows that Lebesgue outer measure satisfies the desired properties (1),(2) and (3) listed at the beginning of this lecture. However, the property (4) is not verified by Lebesgue outer measure as we will present later an example of two *disjoint* sets A, B for which $\mu^*(A \bigcup B) \ne \mu^*(A) + \mu^*(B)$! Of course, we will not want to consider these sets to be normal in order that they are *measurable.* Instead, we will focus our attention on a collection of sets, known as measurable sets, for which the property (4) is valid.

Definition 3.3 *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if for each set $A \subseteq \mathbb{R}$, we have $\mu^*(A) = \mu^*(A \bigcap E) + \mu^*(A \bigcap E^C)$ (Here, $E^C = \mathbb{R} \backslash E$, the complement of E in $\mathbb{R}$). If E is a Lebesgue measurable set, then the* Lebesgue measure *of E, denoted by $\mu(E)$, is defined to be its outer Lebesgue measure $\mu^*(E)$.*

It will not be immediately obvious that the property (4) will be valid for μ. However, we first note that the definition includes sets E that behave at least normally. Basically, E should divide any set A into two disjoint parts, $A \bigcap E$ and $A \bigcap E^C$, whose outer measures add up to give the outer measure of A as expected.

Remark 3.4 Also note that, we have proven in (e) of Theorem 3.2 that $\mu^*(A) \leq \mu^*(A \bigcap E) + \mu^*(A \bigcap E^C)$ so that we need only to show the reverse inequality to establish that the set E is measurable.

The proof of the following elementary facts are left as exercise.

Theorem 3.5 *The collection of measurable sets has the following properties.*

a) $\emptyset$ *and* $\mathbb{R}$ *are measurable.*

b) *If* E *is measurable the so is* E^C.

c) *If* $\mu^*(E) = 0$ *then* E *is measurable.*

We also have

Theorem 3.6 *If* E_1, E_2 *are measurable, then* $E_1 \bigcup E_2$ *and* $E_1 \bigcap E_2$ *are also measurable.*

Proof: Let A be any set in $\mathbb{R}$. Since E_1, E_2 are measurable, we have

$$\begin{aligned} \mu^*(A) &= \mu^*(A \bigcap E_1) + \mu^*(A \bigcap E_1^C) \\ &= \mu^*(A \bigcap E_1) + \mu^*(A \bigcap E_1^C \bigcap E_2) + \mu^*(A \bigcap E_1^C \bigcap E_2^C). \end{aligned}$$

But $A \bigcap (E_1 \bigcup E_2) = (A \bigcap E_1) \bigcup (A \bigcap E_1^C \bigcap E_2)$ and $E_1^C \bigcap E_2^C = (E_1 \bigcup E_2)^C$. Thus, the above implies

$$\mu^*(A) \geq \mu^*(A \bigcap (E_1 \bigcup E_2)) + \mu^*(A \bigcap (E_1 \bigcup E_2)^C),$$

which shows that $E_1 \bigcup E_2$ is measurable. Since $E_1 \bigcap E_2 = (E_1^C \bigcup E_2^C)^C$, the set $E_1 \bigcap E_2$ is also measurable (why?).∎

Theorem 3.7 *If* E *is measurable, then* $E + x_0$ *is measurable.*

Proof: For any $A \subseteq \mathbb{R}$, we compute

$$\begin{aligned} \mu^*(A) &= \mu^*(A \bigcap E) + \mu^*(A \bigcap E^C) \\ (why?) &= \mu^*((A \bigcap E) + x_0) + \mu^*((A \bigcap E^C) + x_0) \\ &= \mu^*((A + x_0) \bigcap (E + x_0)) + \mu^*((A + x_0) \bigcap (E^C + x_0)) \\ &= \mu^*((A + x_0) \bigcap (E + x_0)) + \mu^*((A + x_0) \bigcap (E + x_0)^C). \end{aligned}$$

We replace A by $A - x_0$ in the above and find that

$$\mu^*(A) = \mu^*(A - x_0) = \mu^*(A \bigcap (E + x_0)) + \mu^*(A \bigcap (E + x_0)^C),$$

and it follows that $E + x_0$ is measurable. ∎

Naturally, we should expect that

Theorem 3.8 *Every interval is measurable.*

Proof: Let $a \in \mathbb{R}$. We need only (why?) to show that (a, ∞) is measurable. To this end, we will prove that

$$\mu^*(A \bigcap (a, \infty)) + \mu^*(A \bigcap (-\infty, a]) \leq \mu^*(A). \tag{3.1}$$

This is trivial if $\mu^*(A) = \infty$ so suppose that $\mu^*(A) < \infty$. Let $\varepsilon > 0$. There exists a sequence $\{I_k\}$ of open interval such that $A \subseteq \bigcup_k I_k$ and $\sum_k \ell(I_k) < \mu^*(A) + \varepsilon$. For each k, let $I_k^1 = I_k \bigcap (a, \infty)$ and $I_k^2 = I_k \bigcap (-\infty, a]$. Obviously, $\{I_k^1\}$ and $\{I_k^2\}$ are sequence of intervals that cover $A \bigcap (a, \infty)$ and $A \bigcap (-\infty, a]$, respectively, and $\mu^*(I_k^1) + \mu^*(I_k^2) = \ell(I_k)$. Hence,

$$\mu^*(A \bigcap (a, \infty)) + \mu^*(A \bigcap (-\infty, a]) \leq \sum_k \ell(I_k) < \mu^*(A) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get (3.1). ∎

Our first step in proving the property (4) is the following

Lemma 3.9 *Let $\{E_i\}_{i=1}^n$ be a finite collection of disjoint measurable sets. If $A \subseteq \mathbb{R}$, then*

$$\mu^*(\bigcup_i (A \bigcap E_i)) = \mu^*(A \bigcap (\bigcup_i E_i)) = \sum_i \mu^*(A \bigcap E_i). \tag{3.2}$$

In particular, when $A = \mathbb{R}$, $\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$.

Proof: We prove this by induction. (3.2) is obvious when $n = 1$. Suppose that it holds for some n. Consider $n + 1$ disjoint measurable sets E_i. Since E_{n+1} is measurable,

$$\begin{array}{lcl} \mu^*(A \bigcap (\bigcup_{i=1}^{n+1} E_i)) & = & \mu^*(A \bigcap (\bigcup_{i=1}^{n+1} E_i) \bigcap E_{n+1}) + \mu^*(A \bigcap (\bigcup_{i=1}^{n+1} E_i) \bigcap E_{n+1}^C) \\ & = & \mu^*(A \bigcap E_{n+1}) + \mu^*(A \bigcap (\bigcup_{i=1}^{n} E_i)) \\ (\textit{induction}) & = & \mu^*(A \bigcap E_{n+1}) + \sum_i^n \mu^*(A \bigcap E_i) = \sum_i^{n+1} \mu^*(A \bigcap E_i). \end{array}$$

Thus, (3.2) continues to hold for $n + 1$. ∎

Theorem 3.10 *If $\{E_i\}$ is a sequence of measurable sets, then $\bigcup_i E_i$ and $\bigcap_i E_i$ are also measurable.*

Proof: Let $E = \bigcup_i E_i$, $H_1 = E_1$ and $H_n = E_n - \bigcup_{i=1}^{n-1} E_i$. Then $\{H_i\}$ is a sequence of measurable sets and $E = \bigcup_i H_i$. Let $A \subseteq \mathbb{R}$ and use the previous lemma to see that

$$\mu^*(A) = \mu^*(A \bigcap (\bigcup_{i=1}^n H_i)) + \mu^*(A \bigcap (\bigcup_{i=1}^n H_i)^C) \geq \sum_{i=1}^n \mu^*(A \bigcap H_i) + \mu^*(A \bigcap E^C),$$

because $E^C \subseteq (\bigcup_{i=1}^n H_i)^C$ (why?). Thus, by letting n tend to ∞, we have

$$\mu^*(A) \geq \sum_{i=1}^\infty \mu^*(A \bigcap H_i) + \mu^*(A \bigcap E^C).$$

Since $A \bigcap E = \bigcup_{i=1}^\infty (A \bigcap H_i)$, we have that $\mu^*(A \bigcap E) \leq \sum_{i=1}^\infty \mu^*(A \bigcap H_i)$. This and the above give

$$\mu^*(A \bigcap E) + \mu^*(A \bigcap E^C) \leq \mu^*(A).$$

Hence, E is a measurable set. Since $\bigcap_i E_i = (\bigcup_i E_i^C)^C$, the proof is complete (why?). ∎

Theorem 3.11 *If E_i is an arbitrary sequence of disjoint measurable sets, then $\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$.*

Proof: For each finite n, Lemma 3.9 shows that $\sum_{i=1}^n \mu(E_i) = \mu(\bigcup_{i=1}^n E_i) \leq \mu(\bigcup_{i=1}^\infty E_i)$. Thus, $\sum_{i=1}^\infty \mu(E_i) \leq \mu(\bigcup_{i=1}^\infty E_i)$. The reverse inequality is obvious due to the countable subadditivity. The proof is complete.∎

We have just shown that Lebesgue measure satisfies all four desired properties (1)-(4). Our next question is: What sets are measurable? First of all, we have

Theorem 3.12 *Every open set and every closed set are measurable.*

Proof: Since every open set in $\mathbb{R}$ is the union of countably many open intervals (why?), we can use Theorem 3.8 and Theorem 3.10 and (b) of Theorem 3.5 to get the conclusion. ∎

There are sets that are *not* measurable. We give an example of such set below. This set is not easy to visualize; but it is important to know that there exist such strange sets.

Theorem 3.13 *There exist sets that are not measurable.*

Proof: Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x - y$ is rational. Note that $\sim$ is an equivalence relation (check it!). Thus, this relation divides $\mathbb{R}$ into equivalent classes of the form $\{x + r : r \in Q\}$. Each equivalent class contains a point in $[0, 1]$. Let $E \subset [0, 1]$ be a set that consists of one point from each equivalent class (*Axiom of Choice* is used here). Let $\{r_i\} = Q \bigcap [-1, 1]$ and let $E_i = E + r_i$ for each i. We thus have a sequence of sets.

We will show that

$$[0, 1] \subset \bigcup_i E_i \subset [-1, 2].$$

The second inclusion is obvious (check it). Let's prove the first. For each $x \in [0, 1]$, there is $y \in E$ such that $x \sim y$ and $x - y \in [-1, 1]$. Thus, $x - y = r_i$ for some i. Hence, $x = y + r_i \in E_i$.

Furthermore, the sets E_i are disjoint. Otherwise, if $E_i \bigcap E_j \neq \emptyset$ for $i \neq j$, then there is $y, z \in E$ such that $y + r_i = z + r_j$ which implies that $y \sim z$, a contradiction (as E does not contain two elements of the same class).

Now suppose that E is measurable. Then each E_i is also measurable and $\mu(E_i) = \mu(E)$. Theorem 3.11 yields

$$1 = \ell([0, 1]) \leq \mu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E) \leq \ell([-1, 2]) = 3.$$

The first inequality shows that $\mu(E) \neq 0$ while the second inequality forces $\mu(E) = 0$ (why?). Hence, we have a contradiction. This mean E cannot be measurable. ∎

We should remark that the above proof used only properties (1)-(4) listed at the beginning and nothing special about Lebesgue measure. This means that one cannot construct a measure that satisfies all four properties on all subsets of $\mathbb{R}$.

Also, is the equality $\mu^*(A \bigcup B) = \mu^*(A) + \mu^*(B)$ valid for *all* disjoint sets A, B? The answer is no! Otherwise, we would have from the proof of the above theorem that

$$n\mu^*(E) = \sum_{i=1}^{n} \mu^*(E_i) = \mu^*(\bigcup_{i=1}^{n} E_i) \leq \ell([-1,2] = 3$$

for every n. This forces $\mu^*(E) = 0$. But this implies E is measurable, a contradiction. However, the answer to our question is yes if A, B are disjoint *measurable sets*. That's why we will want to consider only measurable sets.

The above theorems lead us to the following definitions.

Definition 3.14 *A collection $\mathcal{A}$ of sets is an* algebra *if $\emptyset \in \mathcal{A}$, $E^C \in \mathcal{A}$ whenever $E \in \mathcal{A}$, and $\mathcal{A}$ is closed under* finite *unions (and hence finite intersections).*

Definition 3.15 *An algebra $\mathcal{A}$ is a σ-algebra if it is closed under countable unions (and hence countable intersections).*

The previous theorems showed that the collection of Lebesgue measurable sets is a σ-algebra.

It is easy to see that an arbitrary intersection of σ-algebras is also a σ-algebra (prove it!).

Let $\mathcal{B}$ be the intersection of all σ-algebras that contain the open sets. Hence $\mathcal{B}$ is the *smallest* σ-algebra that contains all the open sets. A set in $\mathcal{B}$ is called a *Borel set.* All countable sets, intervals, open sets and closed sets are Borel sets.

From our previous theorems, it is easy to see that every Borel set in $\mathcal{B}$ is measurable (why?).

Two particular classes of Borel will be frequently used. A G_δ set is any set that can be expressed as a countable intersection of open sets. An F_δ set is any set that can be expressed as a countable union of closed sets.

The next theorem relates measure and limit operations and it will be useful once the Lebesgue is defined.

Theorem 3.16 *Let $\{E_n\}$ be a sequence of measurable sets.*

a) *If $E_n \subset E_{n+1}$ for all n and $E = \bigcup_{n=1}^{\infty} E_n$, the $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.*

b) *Suppose that $\mu(E_1) < \infty$. If $E_n \supseteq E_{n+1}$ for all n and $E = \bigcap_{n=1}^{\infty} E_n$, the $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.*

Proof: a) is trivial if $\mu(E_n) = \infty$ for some n (why?). Suppose that $\mu(E_n) < \infty$ for all n. Let $H_1 = E_1$ and $H_n = E_n - E_{n-1}$ for $n \geq 2$. Then $\{H_n\}$ is a sequence of disjoint measurable sets such that $E = \bigcup_{n=1}^{\infty} H_n$. Note that $\mu(H_n) = \mu(E_n) - \mu(E_{n-1}$. We now have,

$$\mu(E) = \sum_{n=1}^{\infty} \mu(H_n) = \lim_{n\to\infty} \left(\sum_{k=2}^{n} (\mu(E_k) - \mu(E_{k-1})) + \mu(E_1) \right) = \lim_{n\to\infty} \mu(E_n).$$

This completes the proof of a). The proof of b) is left as an exercise (Hint: define $\bar{E}_i = E_1 - E_i$, $\bar{E} = E_1 - E$). ∎

The definition of a measurable set may not be useful in other situations and there are other equivalent forms of this definition to be used accordingly. Some of these forms are presented in the following theorem.

Theorem 3.17 *For any set $E \subseteq \mathbb{R}$, the following are equivalent:*

1) *The set E is measurable.*

2) *For each $\varepsilon > 0$, there exists an open set $O \supseteq E$ such that $\mu^*(O - E) < \varepsilon$.*

3) *For each $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $\mu^*(E - K) < \varepsilon$.*

4) *There exists a G_δ set G such that $E \subseteq G$ and $\mu^*(G - E) = 0$.*

5) *There exists a F_δ set F such that $F \subseteq E$ and $\mu^*(E - F) = 0$.*

Proof: We will prove that 1)⇒2)⇒4)⇒1)⇒3)⇒5)⇒1).

1)⇒2): Let E be a measurable set and assume first that $\mu(E) < \infty$. Let $\varepsilon > 0$. By the definition, we can choose a sequence of open intervals $\{I_k\}$ such that $E \subseteq \bigcup_k I_k$ and $\sum_k \ell(I_k) < \mu(E)+\varepsilon$. Let $O = \bigcup_k I_k$ then O is an open set contains E. Since $O = E \bigcup (O-E)$ is a disjoint union of two open sets, we have

$$\mu(O - E) = \mu(O) - \mu(E) \leq \sum_k \ell(I_k) - \mu(E) < \varepsilon.$$

This proves 2) when $\mu(E) < \infty$. Suppose now that $\mu(E) = \infty$ and let $E_n = E \bigcap ([n-1, n) \bigcup (-n, -(n-1)]$ for each integer n. Each of E_n is measurable and $\mu(E_n) < \infty$. Thus, there is an open set O_n such that $E_n \subseteq O_n$ and $\mu(O_n - E_n) < \varepsilon/2^n$. The set $O = \bigcup_n O_n$ is open and $E \subseteq O$. Moreover, since $O - E \subseteq \bigcup_n (O_n - E_n)$, we have

$$\mu(O - E) \leq \sum_n \mu(O_n - E_n) < \sum_n \varepsilon/2^n = \varepsilon.$$

Thus, 1) implies 2).

2)⇒4): For each integer n there is an open set O_n such that $E \subseteq O_n$ and $\mu^*(O_n - E) < 1/n$. Let $G = \bigcap_n O_n$. then G is a G_δ set, $E \subseteq G$ and $\mu^*(G - E) \leq \mu^*(O_n - E) < 1/n$ for all n. This implies that $\mu^*(G - E) = 0$.

4)⇒1): Since $\mu^*(G - E) = 0$, the set $G - E$ is measurable (why?). Thus, $E = G \bigcap (G - E)^C$ is also measurable (why?).

The proof of 1)⇒3)⇒5)⇒1) is similar and will be left as an exercise. ∎

It is important to note that 2) does not state that $\mu^*(O) - \mu^*(E) < \varepsilon$ but $\mu^*(O - E) < \varepsilon$. In fact, by the definition of measure, for any set E (not necessarily measurable) with $\mu^*(E) < \infty$ we can find open set O such that $E \subseteq O$ and $\mu^*(O) - \mu^*(E) < \varepsilon$.

Finally, we introduce here the concept of *inner measure.* But first of all, since a set is open iff it can be expressed as a countable union of open intervals, the outer measure of a set E can be rewritten as

$$\mu^*(E) = \inf\{\mu(O) : E \subseteq O \text{ and } O \text{ is open}\}.$$

This leads to the following definition.

Definition 3.18 *Let $E \subseteq \mathbb{R}$. The inner measure of E is defined by*

$$\mu_*(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is closed}\}.$$

It is clear that $\mu_*(E) \le \mu^*(E)$ and $\mu_*(A) \le \mu_*(B)$ if $A \subseteq B$. We have the following theorem on the basic properties of the inner measure.

Theorem 3.19 *Let A, E be subsets of $\mathbb{R}$.*

a) *Let $\mu^*(E) < \infty$. Then E is measurable if and only if $\mu_*(E) = \mu^*(E)$.*

b) *There exists an F_δ set F such that $F \subseteq E$ and $\mu(F) = \mu_*(E)$.*

c) *If A and E are disjoint, then $\mu_*(A \bigcup E) \ge \mu_*(A) + \mu_*(E)$.*

d) *If E is measurable and $A \subseteq E$, then $\mu(E) = \mu_*(A) + \mu^*(E - A)$.*

Proof: a): Suppose first that E is measurable and let $\varepsilon > 0$. There exists a closed set $K \subseteq E$ such that $\mu(E - K) < \varepsilon$. Thus,

$$\mu^*(E) \ge \mu_*(E) \ge \mu(K) > \mu(E) - \varepsilon = \mu^*(E) - \varepsilon.$$

Since ε is arbitrary, we must have $\mu_*(E) = \mu^*(E)$. Conversely, if $\mu_*(E) = \mu^*(E)$, then for every given $\varepsilon > 0$, we can find a closed set K and an open set G such that $K \subseteq E \subseteq G$ and

$$\mu(K) > \mu_*(E) - \varepsilon/2, \quad \mu(G) < \mu^*(E) + \varepsilon/2.$$

Hence,

$$\mu^*(G - E) \le \mu^*(G - K) = \mu(G - K) = \mu(G) - \mu(K) < \varepsilon.$$

Theorem 3.17 asserts that E is measurable.

b) and c): exercises.

d): For each $\varepsilon > 0$, we can find a closed set $K \subseteq A$ such that $\mu(K) > \mu_*(A) - \varepsilon$. Hence,

$$\mu(E) = \mu(K) + \mu(E - K) > \mu_*(A) - \varepsilon + \mu^*(E - A)$$

and it follows that $\mu(E) \ge \mu_*(A) + \mu^*(E - A)$. By the exercise, we can find a measurable set B such that $E - A \subseteq B \subseteq E$ such that $\mu(B) = \mu^*(E - A)$. Since $E - B \subseteq A$, we have $\mu_*(E - B) \le \mu_*(A)$. Thus,

$$\mu(E) = \mu(B) + \mu(E - B) = \mu^*(E - A) + \mu(E - B) \le \mu^*(E - A) + \mu^*(A).$$

Combining the inequalities, we get $\mu(E) = \mu_*(A) + \mu^*(E - A)$. ∎

Exercises

1. Prove a) and b) of Theorem 3.2.
2. Complete the proof of f) of Theorem 3.2 when the interval I is infinite.
3. Prove that for any set E (not necessarily measurable) with $\mu^*(E) < \infty$ we can find open set O such that $E \subseteq O$ and $\mu^*(O) - \mu^*(E) < \varepsilon$.
4. Let $\{E_n\}$ be a sequence of sets of zero measure. Show that $\mu(\bigcup_n E_n) = 0$.
5. Show that any measurable subset of the nonmeasurable set E described in the proof of Theorem 3.13 must have zero measure.
6. Show that an arbitrary intersection of σ-algebras is also a σ-algebra.
7. Prove part b) of Theorem 3.16.
8. Complete the proof of 1)$\Rightarrow$3)$\Rightarrow$5)$\Rightarrow$1) of Theorem 3.17.
9. Let E be a measurable set with $\mu(E) > 0$. Prove that there is a closed set $K \subseteq E$ such that $\mu(K) > 0$.
10. Let $A \subseteq \mathbb{R}$. Prove that there is a G_δ set G such that $A \subseteq G$ and $\mu(G) = \mu^*(E)$.
11. Prove b) and c) Theorem 3.19.
12. Let $A \subseteq \mathbb{R}$ with $\mu^*(A) < \infty$. Suppose that there is a measurable set B such that $B \subseteq A$ and $\mu(B) = \mu^*(A)$. Prove that A is measurable.

Sample test problems:

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $E \subseteq \mathbb{R}$ be a measurable set with $\mu(E) = 0$. Show that $\mu(f(E)) = 0$. What if f is only locally Lipschitz?
2. Let $A \subseteq \mathbb{R}$ be a measurable set. Define the function $g : [0, \infty) \to [0, \infty)$ by $g(x) = \mu(A \bigcap [-x, x])$.
   (a) Show that g is an increasing function.
   (b) Show that g is Lipschitz. What is its Lipschitz constant?
   (c) What if A is not measurable and $g(x) = \mu^*(A \bigcap [-x, x])$?
3. Let E be the nonmeasurable set described in the lecture notes. Let $A \subset E$. If A is measurable, then show that $\mu(A) = 0$.
4. For any $A \subseteq \mathbb{R}$. Show that there exist a G_δ set G and a F_δ set K such that $K \subseteq A \subseteq G$, and $\mu(G) = \mu^*(A)$ and $\mu(K) = \mu_*(A)$.
5. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of measurable sets such that
$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$
   Show that the set $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ has measure zero.

6. Let $X \subseteq \mathbb{R}$ be a measurable set with $\mu(X) = 1$. Let $A_1, \ldots, A_n$ be measurable subsets of X such that $\sum_{i=1}^{n} \mu(A_i) > n - 1$. Prove that $\mu(\bigcap_{i=1}^{n} A_i) > 0$.

7. Let A, B be two measurable sets. Show that

$$\mu(A \bigcup B) + \mu(A \bigcap B) = \mu(A) + \mu(B).$$

8. Let $A \subseteq \mathbb{R}$ be a measurable set and $B \subseteq \mathbb{R}$ be any set. Assume that A, B are disjoint. Show that

$$\mu^*(A \bigcup B) = \mu(A) + \mu^*(B).$$

9. Let A be any set with $\mu^*(A) < \infty$. Assume that there exists a measurable set $B \subseteq A$ such that $\mu(B) = \mu^*(A)$. Show that A is measurable.