1 Definition

It is necessary to determine the class of functions that will be considered for the Lebesgue integration. We want to guarantee that the sets which arise when working with these functions are measurable. In this context, as well as in many others, the inverse images of sets are more useful than the images of sets. For example, a function is continuous if and only if the inverse image of each open set is open. A variation of this leads to the concept of measurable function.

**Definition 1.1** A function \( f : E \to \mathbb{R} \) is measurable if \( E \) is a measurable set and for each real number \( r \), the set \( \{ x \in E : f(x) > r \} \) is measurable.

As stated in the definition, the domain of a measurable function must be a measurable set. In fact, we will always assume that the domain of a function (measurable or not) is a measurable set unless explicitly mentioned otherwise. From the definition, it is clear that continuous functions and monotone functions are measurable. However, just as there are sets that are not measurable, there are functions that are not measurable.

Let \( E \) be a measurable set with positive measure and let \( A \subseteq E \). The function \( \chi_A \) represents the characteristic function of \( A \). That is,

\[
\chi_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \notin A
\end{cases}
\]

The set \( \{ x \in E : \chi_A(x) > r \} \) is either \( \emptyset, A, \) or \( E \) (check this!). So, \( \chi_A \) is a measurable function if and only if \( A \) is a measurable set.

**Theorem 1.2** Let \( E \) be a measurable set and let \( f : E \to \mathbb{R} \). The following are equivalent:

1. For each real number \( r \), the set \( \{ x \in E : f(x) > r \} \) is measurable.
2. For each real number \( r \), the set \( \{ x \in E : f(x) \geq r \} \) is measurable.
3. For each real number \( r \), the set \( \{ x \in E : f(x) < r \} \) is measurable.
4. For each real number \( r \), the set \( \{ x \in E : f(x) \leq r \} \) is measurable.

**Proof:** To see (1) \( \iff \) (2) we simply note that (check it!)

\[
\{ x \in E : f(x) \geq r \} = \bigcap_{n=1}^{\infty} \{ x \in E : f(x) > r - \frac{1}{n} \},
\]
\[ \{ x \in E : f(x) > r \} = \bigcup_{n=1}^{\infty} \{ x \in E : f(x) \geq r + \frac{1}{n} \}. \]

The other implications are established by taking complements. \(\blacksquare\)

When working with functions, sets of measure zero can often be ignored. A property is said to hold almost everywhere (we will abbreviate it by a.e.) if it holds everywhere except for a set of measure zero. For example, the functions \( f : E \to \mathbb{R} \) and \( g : E \to \mathbb{R} \) are equal almost everywhere if and only if the set \( \{ x \in E : f(x) \neq g(x) \} \) has measure zero. Similarly, \( f \) is continuous a.e. iff the set of discontinuity points has measure 0.

**Theorem 1.3** Let \( f : E \to \mathbb{R} \) be measurable and let \( g : E \to \mathbb{R} \). If \( f = g \) a.e. on \( E \), then \( g \) is measurable.

**Proof:** For \( r \in \mathbb{R} \), let \( A = \{ x \in E : f(x) > r \} \) and \( A = \{ x \in E : g(x) > r \} \). Then \( A \) is measurable and \( A - B, B - A \) are subsets of \( \{ x \in E : f(x) \neq g(x) \} \) so that they have zero measures. Since

\[
B = (B - A) \bigcup (B \cap A) = (B - A) \bigcup (A - (A - B))
\]

is measurable. Thus, \( g \) is measurable. \(\blacksquare\)

2 Operations on measurable functions

**Theorem 2.1** Let \( f : E \to \mathbb{R} \) and \( g : E \to \mathbb{R} \) be measurable functions and let \( k \in \mathbb{R} \). Then the functions \( kf, f + g, |f| \) and \( fg \) are measurable. In addition, the function \( f/g \) is measurable if \( g(x) \neq 0 \) for all \( x \in E \).

**Proof:** We will only prove that \( f + g \) is measurable as the others will be left as exercises. Let \( r \) be any real number. If \( f(x) + g(x) > r \), then there exists a rational number \( q \) such that \( r - g(x) < q < f(x) \). It follows that

\[
\{ x \in E : (f + g)(x) > r \} = \bigcup_{q \in \mathbb{Q}} \{ x \in E : f(x) > q \} \bigcap \{ x \in E : g(x) > r - q \}.
\]

Hence, \( f + g \) is a measurable function. \(\blacksquare\)

The following more general theorem will be useful.

**Theorem 2.2** Let \( f \) and \( g \) be measurable functions defined on \( E \). Let \( F \) be real and continuous function on \( \mathbb{R}^2 \). Then the function \( h(x) = F(f(x), g(x)) \) is measurable.

**Proof:** Let \( G_r = \{(u, v) : F(u, v) > r \} \). Then \( G_r \) is an open subset of \( \mathbb{R}^2 \), and thus we can write it as a countable union of open rectangles. That is, \( G_r = \bigcup_n I_n \) where

\[
I_n = \{(u, v) : a_n < u < b_n, \ c_n < v < d_n \}.
\]
The sets \( \{ x \in E : a_n < f(x) < b_n \} \) and \( \{ x \in E : c_n < g(x) < d_n \} \) are measurable. Thus
\[
\{ x \in E : (f(x), g(x)) \in I_n \} = \{ x \in E : a_n < f(x) < b_n \} \cap \{ x \in E : c_n < g(x) < d_n \}
\]
is also measurable. Hence the same is true for
\[
\{ x \in E : h(x) > r \} = \{ x \in E : (f(x), g(x)) \in G_r \} = \bigcup_n \{ x \in E : (f(x), g(x)) \in I_n \}.
\]
Thus, \( h \) is measurable.

The following theorem shows that many functions we have learned in Real Analysis I are measurable.

**Theorem 2.3** Let \( E \) be measurable. If \( f : E \to \mathbb{R} \) is continuous a.e., then \( f \) is measurable.

**Proof:** Let \( D \) be the set of discontinuities of \( f \). Then \( \mu(D) = 0 \) and all of its subsets are measurable. Let \( r \in \mathbb{R} \) and note that
\[
\{ x \in E : f(x) > r \} = \{ x \in E - D : f(x) > r \} \bigcup \{ x \in D : f(x) > r \}.
\]
We need only show that \( C = \{ x \in E - D : f(x) > r \} \) is measurable (why?). For each \( x \in C \), as \( f \) is continuous at \( x \), we can find \( \delta_x > 0 \) such that if \( y \in V_{\delta_x}(x) \) then \( f(y) > r \) (why?). It is clear that
\[
C = (E - D) \bigcap \bigcup_{x \in C} V_{\delta_x}(x),
\]
which is measurable (why?). This shows that \( f \) is measurable.

Let \( \{ f_n \} \) be a sequence of functions defined on \( E \). We will denote
\[
\sup_n f_n(x) = \sup \{ f_n(x) : n \in \mathbb{N} \} \quad \text{and} \quad \limsup_n f_n(x) = \lim_n \left( \sup_{k \geq n} f_k(x) \right).
\]
It is easy to check (do it!) that
\[
\limsup_n f_n(x) = \lim_n \left( \sup_{k \geq n} f_k(x) \right) = \inf_n \left( \sup_{k \geq n} f_k(x) \right). \tag{2.1}
\]
Similarly, we define
\[
\inf_n f_n(x) = \inf \{ f_n(x) : n \in \mathbb{N} \} \quad \text{and} \quad \liminf_n f_n(x) = \lim_n \left( \inf_{k \geq n} f_k(x) \right). \tag{2.2}
\]
It is easy to verify the following relations
\[
\inf_n f_n(x) = - \sup_n (-f_n(x)) \quad \text{and} \quad \liminf_n f_n(x) = - \limsup_n (-f_n(x)). \tag{2.3}
\]
Theorem 2.4  Let \( \{f_n\} \) be a sequence of measurable functions defined on \( E \). For \( x \in E \), set
\[
g(x) = \sup_n f_n(x) \quad \text{and} \quad h(x) = \limsup_n f_n(x).
\]
Then \( g \) and \( h \) are measurable.

Proof:  We need only to show that \( g \) is measurable since the measurability of \( h \) comes from this fact and the relations (2.1), (2.3) (exercise!). For any \( r \in \mathbb{R} \), we have
\[
\{x \in E : g(x) > r\} = \bigcup_n \{x \in E : f_n(x) > r\}.
\]
This shows that \( g \) is measurable. \( \blacksquare \)

Let \( \{f_n\} \) be a sequence of functions on \( E \) that converges a.e. to a function \( f \). This is to say that the set \( C = \{x \in E : \lim_n f_n(x) \text{ exists}\} \) satisfies \( \mu(E - C) = 0 \). Thus, \( f \) is only defined on \( C \). However, since \( E - C \) has measure zero, it does not matter how the functions \( f_n \) are defined there! As this situation will arise frequently, we will agree upon the convention that, unless otherwise stated, such functions are equal to zero on \( E - C \). Hence, \( f \) is now conveniently defined on the whole set \( E \! \)!

We now have

Corollary 2.5  Let \( f_n \) be a sequence be a sequence of measurable functions defined on \( E \) and \( f : E \to \mathbb{R} \). If \( \{f_n\} \) converges pointwise to \( f \) a.e. on \( E \), then \( f \) is measurable.

The following theorem provides another useful characterization of measurable functions.

Theorem 2.6  \( f : E \to \mathbb{R} \) is measurable iff \( f^{-1}(B) \) is measurable for every Borel set \( B \).

Proof:  Suppose that \( f \) is measurable. Let’s define the following collection
\[
\mathcal{A} = \{A \subseteq \mathbb{R} : f^{-1}(A) \text{ is measurable}\}.
\]
We can easily check the followings (exercise):

- \( \emptyset \in \mathcal{A} \).
- If \( A \in \mathcal{A} \) then \( A^C \in \mathcal{A} \) \( (f^{-1}(A^C) = f^{-1}(\mathbb{R}) - f^{-1}(A) = E - f^{-1}(A)) \).
- If \( \{A_n\} \) is a sequence of sets in \( \mathcal{A} \) then \( \bigcup_n A_n \in \mathcal{A} \) \( (f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)) \).

This shows that \( \mathcal{A} \) is a \( \sigma \)-algebra. Moreover, since \( f \) is measurable, any open interval \( (a, b) \) must be in \( \mathcal{A} \) (That is, \( f^{-1}((a, b)) \) is measurable (why?)). Thus, \( \mathcal{A} \) contains all open sets (why?). From the definition of Borel sets, we see that they must be in \( \mathcal{A} \). Hence, \( f^{-1}(B) \) is measurable for all \( B \in \mathcal{B} \). The converse is trivial (check it!). \( \blacksquare \)
Exercises

1. Let \( f : E \to \mathbb{R} \) and \( A \subseteq \mathbb{R} \). Show that \( f^{-1}(A^C) = E - f^{-1}(A) \). Moreover, if \( \{A_\alpha\} \) is a collection of sets then \( f^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f^{-1}(A_\alpha) \) and \( f^{-1}(\bigcap_\alpha A_\alpha) = \bigcap_\alpha f^{-1}(A_\alpha) \).

2. Let \( A \subseteq \mathbb{R} \). Determine \( r \) such that \( \{x \in E : \chi_A(x) > r\} \) is either (respectively) \( \emptyset \), \( A \), or \( E \).

3. Prove the set equalities in the proof of Theorem 1.2.

4. If \( f \) is measurable, then show that \( f^{-1}((a, b)) \) is measurable for any \( a, b \in \mathbb{R} \).

5. Complete the proof of Theorem 2.1. You cannot use Theorem 2.2!

6. Which of the following functions are measurable. Explain why.

\[
\begin{align*}
  f(x) &= \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}, &
  g(x) &= \begin{cases} x^2 & x \in Q \\ 0 & x \notin Q \end{cases}, &
  h(x) &= \begin{cases} x^3 + x^2 & x \in Q \\ \sin(x) & x \notin Q \end{cases}
\end{align*}
\]

7. For each \( n \in \mathbb{N} \), consider the function

\[
f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n] \\ 0 & x > 1/n \end{cases}
\]

Let \( f(x) = \lim_n f_n(x) \).

(a) Is \( f \) continuous?

(b) Is \( f \) measurable?

8. Let \( f : \mathbb{R} \to \mathbb{R} \). Show that \( f \) is continuous iff \( f^{-1}(A) \) is open for every open set \( A \).

9. Prove (2.1) and (2.3).

10. Show that the function \( h \) defined in Theorem 2.4 is measurable.