4.1 Exponential Functions

- Exponential Growth, graph, features of graphs
- Exponential Decay, graph, features of graphs
- example problems from p. 222

$0.01$ for first day, double each day. For at least 30 days.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pay</td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
<td>0.08</td>
<td>0.16</td>
<td>0.32</td>
<td>0.64</td>
<td>1.28</td>
<td>2.56</td>
<td>5.12</td>
<td>10.24</td>
<td>20.48</td>
<td>40.96</td>
<td>81.92</td>
<td>163.84</td>
</tr>
</tbody>
</table>

0.01, 0.02, 0.04, 0.08, 0.16, 0.32, 0.64, 1.28, 2.56, 5.12, 10.24, 20.48, 40.96, 81.92, 163.84

\[ n = \text{the number of the day.} \]
\[ f(n) = 0.01 \cdot 2^{n-1} \text{ amount paid that day.} \]

\[ f(30) = 0.01 \cdot 2^{29} = 5,368,709.12 \]

Exponential Growth

Simple Form for an exponential function is

\[ f(x) = a^x , \ a > 0 , \ \text{Dom}(f) = (-\infty, \infty), \ \text{Range}(f) = (0, \infty) \]
\[ f(x) = 2^x \quad \text{for} \quad b < 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>(2^{-3} = \frac{1}{8})</td>
</tr>
<tr>
<td>-2</td>
<td>(2^{-2} = \frac{1}{4})</td>
</tr>
<tr>
<td>-1</td>
<td>(2^{-1} = \frac{1}{2})</td>
</tr>
<tr>
<td>0</td>
<td>(2^0 = 1)</td>
</tr>
<tr>
<td>1</td>
<td>(2^1 = 2)</td>
</tr>
<tr>
<td>2</td>
<td>(2^2 = 4)</td>
</tr>
<tr>
<td>3</td>
<td>(2^3 = 8)</td>
</tr>
</tbody>
</table>

\[ 3^3 \cdot 3^{-2} = 3 \cdot \frac{1}{3} \cdot \frac{1}{3} = 3^1 \]

\[ g(x) = \left(\frac{1}{3}\right)^x \quad \text{for} \quad b < 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>((\frac{1}{3})^{-3} = 27)</td>
</tr>
<tr>
<td>-2</td>
<td>((\frac{1}{3})^{-2} = 9)</td>
</tr>
<tr>
<td>-1</td>
<td>((\frac{1}{3})^{-1} = 3)</td>
</tr>
<tr>
<td>0</td>
<td>((\frac{1}{3})^0 = 1)</td>
</tr>
<tr>
<td>1</td>
<td>((\frac{1}{3})^1 = \frac{1}{3})</td>
</tr>
<tr>
<td>2</td>
<td>((\frac{1}{3})^2 = \frac{1}{9})</td>
</tr>
<tr>
<td>3</td>
<td>((\frac{1}{3})^3 = \frac{1}{27})</td>
</tr>
</tbody>
</table>
\[ f(x) = 2^x \]

\[ f(x) = 2^{-x} \]

**swap horizontally**

\[ 2^{-x} = 2^{-(1 \cdot x)} \]

\[ = (2^{-1})^x \]

\[ = \left(\frac{1}{2}\right)^x \]

\[ (3/7)^x = (7/3)^x \]

**base > 1**

\[ \text{increase as you move to the right} \]

**Features of Exponential Growth Function.**

\[ f(x) = a^x \]

- base > 1
- increases from left to right
- graph goes through \((-1, \frac{1}{a}), (0, 1), (1, a)\).

**Features of Exponential Decay Function.**

\[ f(x) = a^x \]

- base < 1
- decreases from left to right
- graph goes through \((-1, \frac{1}{a}), (0, 1), (1, a)\).
Business  The scrap value of a machine is the value of the machine at the end of its useful life. By one method of calculating scrap value, where it is assumed that a constant percentage of value is lost annually, the scrap value is given by

\[ S = C(1 - r)^n, \]

where \( C \) is the original cost, \( n \) is the useful life of the machine in years, and \( r \) is the constant annual percentage of value lost. Find the scrap value for each of the following machines.

42. Original cost, $68,000; life, 10 years; annual rate of value loss, 8%.

\[ S = C(1 - r)^n = 68,000(1 - 0.08)^{10} = 68,000(0.92)^{10} \approx 29,548.41 \]

90% value remaining = 0.92^{10} \approx 43.49%.

43. Original cost, $244,000; life, 12 years; annual rate of value loss, 15%.

\[ S = C(1 - r)^n = 244,000(1 - 0.15)^{12} = 244,000(0.85)^{12} \approx 14,205.03 \]

60% value remaining = 0.85^{12} \approx 30.59%.

44. Use the graphs of \( f(x) = 2^x \) and \( g(x) = 2^{-x} \) (not a calculator) to explain why \( 2^x + 2^{-x} \) is approximately equal to \( 2^x \) when \( x \) is very large.

\[
\begin{align*}
\frac{1}{2^x} & \approx 2^{-x} \\
\text{why} \quad 2^x & \approx 2^x + 2^{-x} \\
\text{when} \ x \ \text{is large?}
\end{align*}
\]
Applications of Exponential Functions

- Equivalency of two models
- Problems from p. 229, do at least one on logistic model

\[ f(x) = A \cdot b^x \quad \text{or} \quad f(x) = A \cdot e^{\lambda x} \]

1. Starting amount: \( b \) (base)
   - \( b \) chosen based on growth or time-speed frame
2. Rate (growth \( r > 0 \), decay \( r < 0 \))
   - \( e \) default exponential value

The number \( e \):

\[ e \approx 2.71828 \ldots \] is irrational

Compounding Interest: \( A = P \left( 1 + \frac{r}{n} \right)^{nt} \) more interest when \( n \) is larger.

Continuous Compounding: (\( n \) very large),

\[ A = P \cdot e^{rt} \]

(Makes calculus easier).
3. **Business** Sales of consumer electronics in recent years are approximated by

\[ f(t) = 79.4 \times 1.1^t \quad (t \geq 3), \]

where \( t = 3 \) corresponds to the year 2003 and \( f(t) \) is in billions of dollars.† Find the sales in the given years.

(a) 2006  
(b) 2009  
(c) If the function remains accurate, what will sales be in 2014? Does this result seem plausible?

\[ f(6) = 79.4 \times 1.1^6 \approx 140.66 \]

\[ f(9) = 79.4 \times 1.1^9 \approx 187.22 \]

\[ f(14) = 79.4 \times 1.1^{14} \approx 301.52 \]

20. **Natural Science** The population of fish in a certain lake at time \( t \) months is given by the function

\[ p(t) = \frac{20,000}{1 + 24(2^{-0.36t})}. \]

(a) Graph the population function from \( t = 0 \) to \( t = 48 \) (a four-year period).

(b) What was the population at the beginning of the period?

(c) Use the graph to estimate the one-year period in which the population grew most rapidly.

(d) When do you think the population will reach 25,000? What factors in nature might explain your answer?

\[ \rho(6) = \frac{20,000}{1 + 24(2^{-0.36 \times 6})} \]

\[ \rho(6) = \frac{20,000}{1 + 24(2^0)} = \frac{20,000}{25} = 800 \]
\[ \rho(1) = \frac{20,000}{1 + 24(2^{-0.36 \cdot 1})} \approx 1015.23 \]

\[ \rho(10) = \frac{2,000}{1 + 24(2^{-0.36 \cdot 10})} \approx 6713.07 \]