7. Linear Programming

7.1 Graphing Linear Inequalities in Two Variables

\[ 3x - 1 < 2 \]
\[ +1 +1 \]
\[ \frac{3x}{3} < \frac{3}{3} \]
\[ x < 1 \]

\[ y = 2x - 1 \]

Graph \( y \geq 3x - 2 \)

Aside: Individual solutions can be found as follows:

\((2, 4)\) test \( \frac{4}{2} \geq 3(2) - 2 \)
\[ 2 \geq 4 \checkmark \]

\((3, 1)\) test \( \frac{1}{1} \geq 3(3) - 2 \)
\[ 1 \geq 7 \times \text{not a solution} \]
1. Graph the boundary: $y = 3x - 2$

2. Decide whether the boundary is solid or dotted:
   - $\geq, \leq$ → solid b/c the points on the boundary are solutions
   - $> , <$ → dotted b/c the points on the boundary are not solutions.

3. Decide which half to shade. $y = 3x - 2$
   - Check $(-2, 2) \rightarrow 2 \geq 3(-2) - 2$
     - $2 \geq -8$ ✅

Graph
\[
\begin{align*}
\{ y &< 2 \\
\{ y &\geq \frac{1}{2}x - 3
\end{align*}
\]

Graph boundaries:
- $y = 2$
- $y = \frac{1}{2}x - 1$

Decide dotted vs. solid:

Feasible solutions:

This corner is not a solution.
Graph
\[ \begin{align*}
    y &\leq \frac{3}{5} x + 2 \\
    y &\leq -\frac{1}{4} x + 3 \\
    x &\geq 1
\end{align*} \]

1. Graph boundaries
   - \( y = \frac{3}{5} x + 2 \), solid
   - \( y = -\frac{1}{4} x + 3 \), solid
   - \( x = 1 \), solid

2. Find the intersections of
   \[ \begin{align*}
   \begin{cases}
   y = \frac{3}{5} x + 2 \\
   x = 1
   \end{cases} &\quad \text{and} \quad \begin{cases}
   y = -\frac{1}{4} x + 3 \\
   x = 1
   \end{cases} \\
   y = \frac{3}{5} (1) + 2 &\quad y = -\frac{1}{4} (1) + 3 \\
   = \frac{8}{5} &\quad = \frac{11}{4} \\
   \approx 1.67 &\quad = 2.75
   \end{align*} \]

3. Shade the region

\((1, \frac{8}{3}) \approx (1, 2.67)\)
7.2 Linear Programming: The Graphical Method

Maximize \( z = x + y \)

on the region \( \begin{cases} y \leq 3 \\ y \geq 1 \\ x \leq 5 \\ x \geq -1 \end{cases} \)

1. Graph boundaries
   - \( y = 3 \), solid
   - \( y = 1 \), solid
   - \( x = 5 \), solid
   - \( x = -1 \), solid

and shade

2. Graph \( z = x + y \) for different values of \( z \).
   - \( z = 0 \rightarrow 0 = x + y \)
   - \( z = 1 \rightarrow 1 = x + y \)
   - \( z = 4 \rightarrow 4 = x + y \)
   - \( z = 6 \rightarrow 6 = x + y \)

at \((5,3)\), \( z = 5 + 3 = 8 \)
at \((-1,1)\), \( z = -1 + 1 = 0 \)

"The Min/Max Value Corner Theorem"

When you are optimizing a linear function of two variables on a bounded region made from linear inequalities, there will always be a min and a max and they will occur at one or more corner points.
A few rough examples:

Maximize \( z = 3x - y \) on the region

\[ \begin{align*}
&y \geq 1 \\
&x \geq 3 \\
&y \leq \frac{5}{3}x + 6
\end{align*} \]

Graph boundaries:
- \( y = 1 \), solid
- \( x = 3 \), solid
- \( y = \frac{5}{3}x + 6 \), solid

\[ z = 3x - y, (3,6), (3,1), (6,1) \]

\[(3,6) \rightarrow z = 3(3) - 6 = 3 \ \text{min} \]
\[(3,1) \rightarrow z = 3(3) - 1 = 8 \]
\[(6,1) \rightarrow z = 3(6) - 1 = 17 \ \text{max} \]