Finishing up 11.1.

\[ \lim_{x \to 3} 2x^2 - x - 1 = \lim_{x \to 3} 2x^2 - \lim_{x \to 3} x - \lim_{x \to 3} 1 \]
\[ = (\lim_{x \to 3} 2)(\lim_{x \to 3} x)(\lim_{x \to 3} 1) - \lim_{x \to 3} x - \lim_{x \to 3} 1 \]
\[ = 2 \cdot 3 \cdot 3 - 3 - 1 \]
\[ = 14 \]

Shortcut:
\[ \lim_{x \to 3} 2x^2 - x - 1 = 2(3)^2 - 3 - 1 = 14 \]

\[ \lim_{x \to 0} \frac{x^4 + 2x}{x^2 - 3} = \lim_{x \to 0} \frac{x^4 + 2x}{x^2 - 3} \]
\[ = \frac{\lim_{x \to 0} (x^4 + 2x)}{\lim_{x \to 0} (x^2 - 3)} \]
\[ = \frac{1 + 2}{1 - 3} \]
\[ = \frac{3}{-2} \]

Shortcut:
\[ \lim_{x \to 0} \frac{x^4 + 2x}{x^2 - 3} = \frac{1 + 2}{1 - 3} \]
\[ = \frac{3}{-2} \]

\[ \lim_{x \to 0} \frac{x^4 - 25}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} \]
\[ = \lim_{x \to 5} x+5 = 10 \]

\[ \lim_{x \to 0} \frac{x^3}{x} = \text{ONE, no limit} \]

Vertical asymptote at \( x = 0 \).

(might not be in the book)

\[ \lim_{x \to a} e^x = e^a, \quad \lim_{x \to a} \ln(x) = \ln(a) \]
\[ \text{at } (1,0) \quad \lim_{x \to 0^+} \ln(x) = -\infty \]
\[
\lim_{x \to 0^-} f(x) = 4 \\
\lim_{x \to 0^+} f(x) = 2 \\
\lim_{x \to -\infty} f(x) = \text{DNE} \\
\lim_{x \to -1^-} f(x) = -\infty \\
\lim_{x \to -1^+} f(x) = +\infty \\
\lim_{x \to -1} f(x) = \text{DNE}
\]

\[
\lim_{x \to 0^-} f(x) = 2 \\
\lim_{x \to 0^+} f(x) = 2 \\
\lim_{x \to 0} f(x) = 2 \neq f(0)
\]
11.3 Rates of Change

average rate of change of distance function vs. any function

geometric meaning of average rate of change

instantaneous rate of change (next page)

We begin the discussion with a familiar situation. A driver makes the 168-mile trip from Cleveland to Columbus, Ohio, in 3 hours. The following table shows how far the driver has traveled from Cleveland at various times:

<table>
<thead>
<tr>
<th>Time (in hours)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (in miles)</td>
<td>0</td>
<td>22</td>
<td>52</td>
<td>86</td>
<td>118</td>
<td>148</td>
<td>168</td>
</tr>
</tbody>
</table>

If \( f \) is the function whose rule is

\[ f(x) = \text{distance from Cleveland at time } x, \]

What is the average speed between \( x = 1 \) and \( x = 2 \)?

\[
\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{118 - 52}{2 - 1} = \frac{66}{1} = 66 \text{ mph}
\]

What is the average speed between \( x = 0 \) and \( x = 3 \)?

\[
\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{168 - 0}{3 - 0} = \frac{168}{3} = 56 \text{ mph}
\]
26. \( f(x) = x^2 + x \)

\[ f(x) = x^2 + x \]

Find the Average Rate of Change from \( x = 1 \) to \( x = 5 \), \([1, 5] \)

\[ ARoC = \frac{\text{change in output}}{\text{change in input}} \]

\[ = \frac{f(5) - f(1)}{5 - 1} \]

\[ = \frac{30 - 2}{4} \]

\[ = 7 \]

Find \( ARoC \) from \( x = 1 \) to \( x = 4 \), \([1, 4] \)

\[ ARoC = \frac{f(4) - f(1)}{4 - 1} \]

\[ = \frac{20 - 2}{3} \]

\[ = 6 \]

Find \( ARoC \) from \( x = 1 \) to \( x = 2 \), \([1, 2] \)

\[ ARoC = \frac{f(2) - f(1)}{2 - 1} \]

\[ = \frac{6 - 2}{1} \]

\[ = 4 \]

Find \( ARoC \) on \([1, 1.5]\)

\[ ARoC = \frac{f(1.5) - f(1)}{1.5 - 1} \]

\[ = \frac{1.75 + 1.5 - 2}{0.5} \]

\[ = 3.5 \]

Summary

<table>
<thead>
<tr>
<th>interval length</th>
<th>( ARoC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.5</td>
</tr>
</tbody>
</table>
“ARoC” at \( x = 1 \) is 3.

This is called the **Instantaneous Rate of Change (IRoC)**. 

\[
f(x) = x^2 + x
\]

**IRoC of** \( f(x) \) **at** \( x = 1 \) **is**

\[
\lim_{{h \to 0}} \frac{f(1+h) - f(1)}{h} = \lim_{{h \to 0}} \frac{(1+h)^2 + (1+h) - 1^2 + 1}{h} = \lim_{{h \to 0}} \frac{h^2 + 3h + 2}{h} = \lim_{{h \to 0}} (h + 3) = 3
\]

The **IRoC of** \( f(x) \) **at** \( x = a \) **is**

\[
\lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h} = \lim_{{x \to a}} \frac{f(x) - f(a)}{x - a}
\]

**Find the IRoC of** \( g(x) = \sqrt{x+1} \) **at** \( x = 1 \)

\[
\text{IRoC} = \lim_{{h \to 0}} \frac{f(1+h) - f(1)}{h} = \lim_{{h \to 0}} \frac{\sqrt{1+h+1} - \sqrt{2}}{h} = \lim_{{h \to 0}} \frac{\sqrt{h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{h+2} + \sqrt{2}}{\sqrt{h+2} + \sqrt{2}} = \lim_{{h \to 0}} \frac{h+2 - 2}{h(\sqrt{h+2} + \sqrt{2})} = \lim_{{h \to 0}} \frac{h}{h(\sqrt{h+2} + \sqrt{2})} = \lim_{{h \to 0}} \frac{1}{\sqrt{h+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
\]

\[
f(1) = \sqrt{1+1} = \sqrt{2}
\]

\[
f(1+h) = \sqrt{(1+h)+1} = \sqrt{h+2}
\]