



Extrapolation of difference methods in option valuation

Armando Arciniega *, Edward Allen

Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA

Abstract

In the present investigation, the fully implicit and Crank–Nicolson difference schemes for solving option prices are analyzed. It is proved that the error expansions for the difference methods have the correct form for applying Richardson extrapolation to increase the order of accuracy of the approximations. The difference methods are applied to European, American, and down-and-out knock-out call options. Computational results indicate that Richardson extrapolation significantly decreases the amount of computational work (by as much as a factor of 16) in estimation of option prices.

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1. Introduction

Numerical methods in option valuation have been investigated by many researchers (see for example [1–3], and the references therein). An *option* is a security giving the right to buy or sell an asset, subject to certain conditions, within a specified period of time. In this paper, we consider three different types of options. An *American option* is one that can be exercised at any time up to the date the option expires. A *European option* can be exercised only on the expiration date. A *down-and-out knock-out option* is like a standard option with the proviso that if the asset value ever reaches a barrier or knock-out price

* Corresponding author.

E-mail address: aarcinie@math.ttu.edu (A. Arciniega).

below the current asset price during the option's lifetime the option gets canceled. If the asset price always remains above the barrier, the payout is the same as that of a standard option (see [2]). Throughout the present investigation, we will consider call options, although the computational approach is also valid for put options.

There are several numerical methods for approximation of call option valuation which have been developed, analyzed, and computationally tested. For example, Boyle and Lau [2] make a modification to the standard binomial method to obtain a fairly accurate method for barrier European call options. Their numerical results are compared with results given by the Black–Scholes formula. Broadie and Detemple [3] develop lower and upper bounds on the prices of American call (and put) options written on a dividend-paying asset. They also introduce a modification of the binomial method, but it is quite different from the modification of Boyle and Lau. In their modification, at the time step just before the option maturity, the Black–Scholes formula replaces the usual continuation value. They call this modification BBS for binomial with Black–Scholes modification. Then, Richardson extrapolation is added to the BBS method to obtain the BBSR method. Their numerical results are compared with true values based on the binomial method with 15,000 time steps.

In this paper, we consider numerical solution of the arbitrage-free partial differential equation (PDE) described in the paper by Andersen and Rupert [1]. In the present investigation, we focus on the implicit and Crank–Nicolson difference schemes to approximate the solution, and then apply Richardson extrapolation to achieve a more accurate approximation. These difference methods have excellent convergence properties and are applicable to arbitrary option types, although we only consider the option types described above. Suppose, for example, that the calculated option value using the Crank–Nicolson difference scheme is $V_{h,\tau}$ and $V_{h,\tau} = V + uh^2 + z\tau^2 + O(h^\alpha + \tau^\beta)$ where V is the exact value, h and τ are interval widths in price and time, respectively, u and z are independent of h and τ , and $\alpha, \beta > 2$. In Richardson extrapolation, after computing the difference approximations $V_{h,\tau}$ and $V_{(h/2),(\tau/2)}$, a new estimate $\hat{V}_{(h/2),(\tau/2)}$ is calculated using the formula $\hat{V}_{(h/2),(\tau/2)} = (4V_{(h/2),(\tau/2)} - V_{h,\tau})/3$. As a result, the low order terms in the error are eliminated and one obtains that $\hat{V}_{(h/2),(\tau/2)} = V + O(h^\alpha + \tau^\beta)$. Richardson extrapolation is therefore a very, effective, and efficient technique for increasing the accuracy of the difference approximations.

The paper is organized as follows. First, we develop the convergence theory for the difference methods. Specifically, we prove that the error expansion has the correct form for applying Richardson extrapolation. Then we apply the method to the three types of call options under consideration. Our computational results indicate that Richardson extrapolation can significantly increase the accuracy of the option price estimates.

2. A no-arbitrage partial differential equation for option value

An underlying asset is assumed whose price S satisfies, for time $0 \leq t \leq t_{\max}$, the Ito stochastic differential equation:

$$dS_t = S_t[(r(t) - \gamma(t)) dt + \sigma(S_t, t) dW_t], \quad (2.1)$$

where W_t is the standard Wiener process, $\gamma(t)$ is the dividend rate, $r(t)$ is the rate of interest, and σ is the volatility. Let $V(S, t)$ be the value of an option on the asset with maturity $T \in [0, t_{\max}]$ and payout function g . The general pricing formula for arbitrary options is the *no-arbitrage PDE* (see [1]):

$$\frac{\partial V(S, t)}{\partial t} + \frac{1}{2} \sigma^2(S, t) S^2 \frac{\partial^2 V(S, t)}{\partial S^2} + [r(t) - \gamma(t)] S \frac{\partial V(S, t)}{\partial S} = r(t) V(S, t), \quad (2.2)$$

$t \in [0, T]$, where the initial and boundary conditions for this equation are described below. We point out that this equation is solved backwards in time from $t = T$ to $t = 0$. Here t represents the current time, S_t is the current stock price, T is the option maturity, r is the interest rate, γ is the dividend rate, and σ is the volatility. Also, $V : \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}$ denotes the price of the option at time t when the stock price is S . The initial condition for (2.2) is given by:

$$V(S_T, T) = g(S_T), \quad (2.3)$$

where $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ is the payout function.

In the present investigation, several kinds of call options are considered, although the numerical procedures described are applicable to put options and additional types of call options. Specifically considered here are European, American, and down-and-out knock-out call options. For European call options the payout function has the form:

$$g(S_T) = \max(S_T - K, 0), \quad (2.4)$$

where $K > 0$ is the strike price of the option. For American call options, the payout function has the same form as for European call options, i.e., (2.4) holds. However, the early exercise feature for American call options can be incorporated by imposing the constraint:

$$V(S_t, t) \geq \max(S_t - K, 0) \quad \text{for } 0 \leq t \leq T. \quad (2.5)$$

Down-and-out knock-out options have the identical payout function g as American and European options. Reasonable boundary conditions for American and European options are (see [1]):

$$V(0, t) = 0, \quad 0 \leq t \leq T \quad (2.6)$$

and

$$V(S_{\max}, t) = S_{\max} \exp\left(-\int_t^T \gamma(u) du\right) - K \exp\left(-\int_t^T r(u) du\right) \quad (2.7)$$

for $0 \leq t \leq T$ and $S_{\max} \gg K$, although, for S_{\max} sufficiently high in relation to S_0 and K , the boundary condition at S_{\max} does not significantly affect the value of $V(S_0, 0)$. Down-and-out knock-out options have the additional condition:

$$V(S_t, t) = 0, \quad 0 \leq S_t \leq S_K, \quad 0 \leq t \leq T, \quad (2.8)$$

where S_K is the knock-out level. In the special case of a constant volatility $\sigma(S_t, t) = \theta > 0$, the solution of (2.2), subject to (2.4), for the European call option is given by the (extended) Black–Scholes formula (see [1]):

$$V_{BS}(S_t, t; T, K, \theta) = S_t \exp\left[-\int_t^T \gamma(u) du\right] N(d_+) - \exp\left[-\int_t^T r(u) du\right] KN(d_-), \quad (2.9)$$

$$t \in [0, T), \quad d_{\pm} = \frac{\ln(S_t/K) + \int_t^T [r(u) - \gamma(u)] du}{\theta\sqrt{T-t}} \pm \frac{1}{2}\theta\sqrt{T-t},$$

where $N(\cdot)$ is the standard commulative normal distribution function.

To simplify the analysis of Eq. (2.2), let $\hat{t} = T - t$. Then, an equation is obtained that is solved forwards in time from $\hat{t} = 0$ to $\hat{t} = T$. Eq. (2.2) becomes:

$$\begin{aligned} & -\frac{\partial V(S, T - \hat{t})}{\partial \hat{t}} + \frac{1}{2}\sigma^2(S, T - \hat{t})S^2 \frac{\partial^2 V(S, T - \hat{t})}{\partial S^2} \\ & + [r(T - \hat{t}) - \gamma(T - \hat{t})]S \frac{\partial V(S, T - \hat{t})}{\partial S} \\ & = r(T - \hat{t})V(S, T - \hat{t}) \quad \text{for } 0 \leq \hat{t} \leq T. \end{aligned} \quad (2.10)$$

Now, it is useful to let $x = S$ and $H(x, t) = V(S, T - t)$ where \hat{t} is replaced by t for ease of notation. Then the governing equation becomes:

$$\begin{aligned} & \frac{\partial H(x, t)}{\partial t} + \frac{1}{2}a(x, t) \frac{\partial^2 H(x, t)}{\partial x^2} + b(x, t) \frac{\partial H(x, t)}{\partial x} \\ & = q(t)H(x, t) \quad \text{for } 0 \leq t \leq T, \end{aligned} \quad (2.11)$$

where $b(x, t) = [-r(T - t) + \gamma(T - t)]x$, $a(x, t) = -x^2\sigma^2(x, T - t)$ and $q(t) = -r(T - t)$. Notice that the initial condition (2.4) has the form:

$$h(x) \equiv H(x, 0) = \begin{cases} 0 & \text{if } 0 \leq x \leq K, \\ x - K & \text{if } K \leq x \leq \hat{K}, \end{cases} \quad (2.12)$$

where \widehat{K} is chosen sufficiently large so that the solution $H(x_T, T)$ is unaffected by the boundary conditions at $x = \widehat{K}$. In particular, it is assumed that the boundary conditions are smooth and compatible with the initial condition. Also, by (2.6) the boundary condition at $x = 0$ is zero:

$$H(0, t) = 0. \tag{2.13}$$

Notice that the initial data is not smooth. To derive an error expansion for the difference schemes for approximation of (2.11), a smooth function will be used as an approximation to the initial condition. It is shown in the Appendix B that the error in using this approximation can be made negligibly small. Therefore, we can let $\hat{h}(x)$ be an approximation to $h(x)$, where $\hat{h}(x)$ is given by:

$$\hat{h}(x) = \sum_{n=1}^{\infty} b_n e^{-(\pi n)^2 \hat{a}} \cos\left(\frac{\pi n x}{\widehat{K}}\right) \quad \text{for some constant } \hat{a} > 0, \tag{2.14}$$

where the coefficients $\{b_n\}_{n=1}^{\infty}$ are chosen so that:

$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{\pi n x}{\widehat{K}}\right) = \begin{cases} 0 & \text{if } 0 \leq x \leq K, \\ x - K & \text{if } K \leq x \leq \widehat{K}. \end{cases}$$

Thus $h(x) \approx \hat{h}(x)$ for “small” \hat{a} . Notice that \hat{h} is infinitely differentiable; in particular,

$$\begin{aligned} \hat{h}''(x) &= \sum_{n=1}^{\infty} -b_n \left(\frac{\pi n}{\widehat{K}}\right)^2 e^{-(\pi n)^2 \hat{a}} \cos\left(\frac{\pi n x}{\widehat{K}}\right), \\ \hat{h}'''(x) &= \sum_{n=1}^{\infty} +b_n \left(\frac{\pi n}{\widehat{K}}\right)^4 e^{-(\pi n)^2 \hat{a}} \cos\left(\frac{\pi n x}{\widehat{K}}\right). \end{aligned}$$

For small \hat{a} , $\hat{h}(x)$ is a good approximation to $h(x)$ and we can replace $h(x)$ by $\hat{h}(x)$ with only a small error in the solution as explained in the Appendix B. Thus, we consider the following problem:

$$\frac{\partial H(x, t)}{\partial t} + \frac{1}{2} a(x, t) \frac{\partial^2 H(x, t)}{\partial x^2} + b(x, t) \frac{\partial H(x, t)}{\partial x} = q(t)H(x, t), \tag{2.15}$$

$$H(x, 0) = \hat{h}(x). \tag{2.16}$$

The difference schemes can be applied directly to (2.15) and (2.16). However, to derive an error expansion for application of Richardson extrapolation, it is useful to have a homogeneous initial condition. To make the initial condition homogeneous, let

$$\widehat{H}(x, t) = H(x, t) - \hat{h}(x).$$

Then (2.15) and (2.16) become:

$$\begin{aligned} \frac{\partial \widehat{H}(x, t)}{\partial t} + \frac{1}{2} a(x, t) \frac{\partial^2 \widehat{H}(x, t)}{\partial x^2} + b(x, t) \frac{\partial \widehat{H}(x, t)}{\partial x} \\ = q(t) \widehat{H}(x, t) + f(x, t), \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} f(x, t) &= q(t) \widehat{h}(x) - b(x, t) \widehat{h}'(x) - \frac{1}{2} a(x, t) \widehat{h}''(x), \\ \widehat{H}(x, 0) &= 0. \end{aligned} \quad (2.18)$$

For the boundary conditions at \widehat{K} , we assume:

$$\widehat{H}(\widehat{K}, t) = 0, \quad (2.19)$$

where \widehat{K} is chosen sufficiently large so that the boundary at \widehat{K} has negligible effect on the solution, as mentioned earlier. Thus, the problem under investigation becomes:

$$\frac{\partial \widehat{H}(x, t)}{\partial t} + \frac{1}{2} a(x, t) \frac{\partial^2 \widehat{H}(x, t)}{\partial x^2} + b(x, t) \frac{\partial \widehat{H}(x, t)}{\partial x} = q(t) \widehat{H}(x, t) + f(x, t), \quad (2.20)$$

$$\widehat{H}(x, 0) = 0, \quad (2.21)$$

$$\widehat{H}(0, t) = \widehat{H}(\widehat{K}, t) = 0. \quad (2.22)$$

We consider the unique solution of (2.20)–(2.22). Let Q be the rectangle $(0, \widehat{K}) \times (0, T)$, and \overline{Q} its closure. Define the Banach space $H^{l, (l/2)}(\overline{Q})$ as the set of functions that are continuous in \overline{Q} , together with all derivatives $D_t^r D_x^s$ for $2r + s < l$. We then have the following result.

Theorem 2.1. *Suppose that $a, b, f \in H^{k, (k/2)}(\overline{Q})$. Then (2.20)–(2.22) has a unique solution $\widehat{H}(x, t)$ in $H^{k+2, (k/2)+1}(\overline{Q})$.*

Proof. The proof follows directly from Theorem 5.2 in [4]. Thus, for example, if $a, b, f \in H^{4,2}(\overline{Q})$, then $\widehat{H} \in H^{6,3}(\overline{Q})$. \square

In the next section, numerical solution of (2.20)–(2.22) is considered using finite-difference methods. It is shown that the error expansions have the correct form for applying Richardson extrapolation.

3. Error analyses of finite-difference methods for option value

To construct the finite-difference methods, a difference net is first constructed. We fix an integer $M \geq 2$ and let $\tau = (T/M)$. Then, we choose a uniform net in the time variable t :

$$\bar{\omega}_\tau = \{t_j = j\tau; j = 0, 1, \dots, M\},$$

$$\omega_\tau = \{t_j = j\tau; j = 1, 2, \dots, M\}.$$

We also fix an integer $N \geq 2$ and let $h = (\hat{K}/N)$. Then, we choose a uniform net in the variable x :

$$\bar{\omega}_h = \{x_i = ih; i = 0, 1, \dots, N\},$$

$$\omega_h = \{x_i = ih; i = 1, 2, \dots, N - 1\}.$$

We form two-dimensional rectangular nets as the Cartesian products:

$$\bar{Q}_h^\tau = \bar{\omega}_h \times \bar{\omega}_\tau, \quad Q_h^\tau = \omega_h \times \omega_\tau.$$

Then, we seek an approximate solution at the points of \bar{Q}_h^τ and construct the difference equations at the points of the net Q_h^τ . The method of finite differences does not require that the mesh be uniform. However, uniformity of the mesh is assumed throughout the present investigation. We replace Eq. (2.20) by the following difference scheme:

$$\begin{aligned} & \frac{\hat{H}(x, t + \tau) - \hat{H}(x, t)}{\tau} + b(x, t) \left[\frac{\hat{H}(x + h, t) - \hat{H}(x - h, t)}{2h} \right] (1 - \Theta) \\ & + b(x, t + \tau) \left[\frac{\hat{H}(x + h, t + \tau) - \hat{H}(x - h, t + \tau)}{2h} \right] \Theta \\ & + \frac{1}{2} a(x, t) \left[\frac{\hat{H}(x + h, t) - 2\hat{H}(x, t) + \hat{H}(x - h, t)}{h^2} \right] (1 - \Theta) \\ & + \frac{1}{2} a(x, t + \tau) \left[\frac{\hat{H}(x + h, t + \tau) - 2\hat{H}(x, t + \tau) + \hat{H}(x - h, t + \tau)}{h^2} \right] \Theta \\ & = q(t)\hat{H}(x, t) + f(x, t), \quad \text{where } \Theta \in [0, 1]. \end{aligned} \tag{3.23}$$

The parameter $\Theta \in [0, 1]$ determines the type of numerical method. If $\Theta = 0$, the x derivatives are evaluated at time t , and the difference scheme becomes the *explicit scheme*. If $\Theta = 1$, the x derivatives are evaluated one time-step ahead, at $t + \tau$, and the scheme becomes the *fully implicit scheme*. When $\Theta = 1/2$, the scheme becomes the *Crank–Nicolson scheme*. In this paper we consider the cases when $\Theta = 1, 1/2$. It is proved theoretically, and seen computationally, that the Crank–Nicolson scheme converges faster than the fully

implicit scheme. In practice the difference schemes can be applied directly to (2.2). However, to prove an error expansion of the correct form for applying Richardson extrapolation, the equivalent problem of applying the difference methods to (2.20) is investigated.

First, the case $\Theta = 1$ (fully implicit) is considered in detail. It is shown that this method has an error expansion suitable for application of Richardson extrapolation. Using the fully implicit scheme, Eq. (3.23) reduces to:

$$\begin{aligned} &\widehat{H}_t^\tau(x, t) + b(x, t)\widehat{H}_x^\tau(x, t) + \frac{1}{2}a(x, t)\widehat{H}_{xx}^\tau(x, t) \\ &= q(t)\widehat{H}^\tau(x, t) + f(x, t) \quad \text{on } \mathcal{Q}_h^\tau, \end{aligned} \tag{3.24}$$

where the notation \widehat{H}_x , \widehat{H}_{xx} and \widehat{H}_t are defined in Appendix A and \widehat{H}^τ is, for brevity, written without the second index h . The initial and boundary conditions are replaced by the equations:

$$\widehat{H}^\tau(x, 0) = 0, \quad x \in \omega_h, \tag{3.25}$$

$$\widehat{H}^\tau(0, t) = \widehat{H}^\tau(\widehat{K}, t) = 0, \quad t \in \omega_\tau. \tag{3.26}$$

By using techniques similar to those used in [7], the following estimate characterizes the stability of the numerical solution:

$$\max_{x \in \omega_h} |\widehat{H}^\tau(x, t_{j+1})| \leq \max_{x \in \omega_h} |\widehat{H}^\tau(x, 0)| + \sum_{j=0}^j \tau \max_{x \in \omega_h} |f(x, t_j)|,$$

where it is assumed that $q(t) \leq 0$, $a(x, t) \leq 0$, $h|b(x, t)| \leq |a(x, t)|$, $|q(t)|\tau < 1$, and $1 + \tau q(t) + (1/2)a(x, t)(\tau/h^2) > 0$, for all x, t . From this estimate the following corollary holds:

$$\max_{\overline{\mathcal{Q}}_h} |\widehat{H}^\tau| \leq T \max_{\mathcal{Q}_h} |f|. \tag{3.27}$$

It is now shown in the following theorem that the error in approximating \widehat{H} by \widehat{H}^τ has the correct form for applying Richardson extrapolation.

Theorem 3.1. *Suppose that the conditions of Theorem 2.1 hold for (2.20)–(2.22), with $k = 4$, and inequality (3.27) holds. Then, the solution of the implicit difference equation (3.24)–(3.26) can be written as:*

$$\widehat{H}^\tau = \widehat{H} + h^2v + \tau w + (h^4 + \tau^2)\eta^\tau \quad \text{on } \overline{\mathcal{Q}}_h^\tau, \tag{3.28}$$

where the functions v and w are continuous on $\overline{\mathcal{Q}}$ and are independent of τ and h , and the net function η^τ is uniformly bounded:

$$|\eta^\tau| \leq c_1 \quad \text{on } \overline{\mathcal{Q}}_h^\tau. \tag{3.29}$$

Proof. The proof is similar in structure to the proof of Theorem 1.2 of Section 5.1 of [6]. Choose the function v as a solution of the following problem:

$$\frac{\partial v}{\partial t} + b \frac{\partial v}{\partial x} + \frac{1}{2} a \frac{\partial^2 v}{\partial x^2} - qv = g_1 \quad \text{on } Q, \tag{3.30}$$

where

$$g_1 = -\frac{1}{6} b \frac{\partial^3 \widehat{H}}{\partial x^3} - \frac{1}{24} a \frac{\partial^4 \widehat{H}}{\partial x^4},$$

$$v(x, 0) = 0, \quad x \in [0, \widehat{K}], \tag{3.31}$$

$$v(0, t) = v(\widehat{K}, t) = 0, \quad t \in [0, T]. \tag{3.32}$$

Since $g_1 \in H^{2,1}(\overline{Q})$, one can readily see that the conditions in Theorem 2.1 are satisfied, where the number k should be replaced by 2 (see [5,6]). Thus the function $v \in H^{4,2}(\overline{Q})$. Choose the function w as the solution of the following problem:

$$\frac{\partial w}{\partial t} + b \frac{\partial w}{\partial x} + \frac{1}{2} a \frac{\partial^2 w}{\partial x^2} - qw = g_2 \quad \text{on } Q, \tag{3.33}$$

where

$$g_2 = -\frac{1}{2} \frac{\partial^2 \widehat{H}}{\partial t^2},$$

$$w(x, 0) = 0, \quad x \in [0, \widehat{K}], \tag{3.34}$$

$$w(0, t) = w(\widehat{K}, t) = 0, \quad t \in [0, T]. \tag{3.35}$$

Since $g_2 \in H^{2,1}(\overline{Q})$ and the conditions of Theorem 2.1 are satisfied, we have that $w \in H^{4,2}(\overline{Q})$. The functions \widehat{H}^τ , \widehat{H} , v , w are defined unambiguously at the net points of Q_h^τ . Therefore, one considers (3.28) as defining the net function η^τ . That is,

$$(h^4 + \tau^2)\eta^\tau = \widehat{H}^\tau - \widehat{H} - h^2v - \tau w \quad \text{on } \overline{Q}_h^\tau.$$

In order to complete the proof of the theorem, we must obtain the estimate (3.29). We proceed in the following manner. Substituting (3.28) into (3.24) for \widehat{H}^τ , we obtain:

$$\begin{aligned} & \widehat{H}_t + h^2v_t + \tau w_t + (h^4 + \tau^2)\eta_t^\tau + b[\widehat{H}_x + h^2v_x + \tau w_x + (h^4 + \tau^2)\eta_x^\tau] \\ & + \frac{1}{2} a[\widehat{H}_{xx} + h^2v_{xx} + \tau w_{xx} + (h^4 + \tau^2)\eta_{xx}^\tau] \\ & = q[\widehat{H} + h^2v + \tau w + (h^4 + \tau^2)\eta^\tau] + f \quad \text{on } Q_h^\tau. \end{aligned} \tag{3.36}$$

Using Lemmas A.1–A.3 of the Appendix A we have the following expansions:

$$\begin{aligned} \widehat{H}_t &= \frac{\partial \widehat{H}}{\partial t} + \frac{\tau}{2} \frac{\partial^2 \widehat{H}}{\partial t^2} + \tau^2 \xi_1, \\ v_t &= \frac{\partial v}{\partial t} + \tau \xi_2, \\ w_t &= \frac{\partial w}{\partial t} + \tau \xi_3, \\ \widehat{H}_{\dot{x}\dot{x}} &= \frac{\partial^2 \widehat{H}}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 \widehat{H}}{\partial x^4} + h^4 \xi_4, \\ v_{\dot{x}\dot{x}} &= \frac{\partial^2 v}{\partial x^2} + h^2 \xi_5, \\ w_{\dot{x}\dot{x}} &= \frac{\partial^2 w}{\partial x^2} + h^2 \xi_6, \\ \widehat{H}_{\dot{x}} &= \frac{\partial \widehat{H}}{\partial x} + \frac{h^2}{6} \frac{\partial^3 \widehat{H}}{\partial x^3} + h^4 \xi_7, \\ v_{\dot{x}} &= \frac{\partial v}{\partial x} + h^2 \xi_8, \\ w_{\dot{x}} &= \frac{\partial w}{\partial x} + h^2 \xi_9, \end{aligned}$$

where $|\xi_i| \leq c_i, i = 1, \dots, 9$ and the constants c_i are independent of the points of Q_h^τ and h, τ . Substituting the above expansions into (3.36) we obtain:

$$\begin{aligned} &\frac{\partial \widehat{H}}{\partial t} + \frac{1}{2} a \frac{\partial^2 \widehat{H}}{\partial x^2} + b \frac{\partial \widehat{H}}{\partial x} + h^2 \left(\frac{\partial v}{\partial t} + b \frac{\partial v}{\partial x} + \frac{1}{2} a \frac{\partial^2 v}{\partial x^2} - qv \right) \\ &+ \tau \left(\frac{\partial w}{\partial t} + b \frac{\partial w}{\partial x} + \frac{1}{2} a \frac{\partial^2 w}{\partial x^2} - qw \right) + (h^2 \tau + \tau^2) \xi_{10} + (h^4 + \tau^2) \eta_i^\tau \\ &+ (h^4 + h^2 \tau) \xi_{11} + b(h^4 + \tau^2) \eta_{\dot{x}}^\tau + \frac{1}{2} a(h^4 + \tau^2) \eta_{\dot{x}\dot{x}}^\tau \\ &= h^2 \left(-\frac{1}{6} b \frac{\partial^3 \widehat{H}}{\partial x^3} - \frac{1}{24} a \frac{\partial^4 \widehat{H}}{\partial x^4} \right) + \tau \left(-\frac{1}{2} \frac{\partial^2 \widehat{H}}{\partial t^2} \right) + q\widehat{H} \\ &+ q(h^4 + \tau^2) \eta_i^\tau + f \quad \text{on } Q_h^\tau. \end{aligned} \tag{3.37}$$

Here, $|\xi_{10}| \leq c_1 + c_2 + c_3$, and $|\xi_{11}| \leq |a|(c_4 + c_5 + c_6) + |b|(c_7 + c_8 + c_9)$. The terms in h^0 and τ^0 cancel, since the function \widehat{H} satisfies Eq. (2.20). Also, the terms with coefficient h^2 on both sides of Eq. (3.37) cancel, since the function v satisfies Eq. (3.30). Finally, the terms with coefficient τ cancel, since w satisfies Eq. (3.33). Rewriting the terms in formula (3.37) we obtain:

$$\begin{aligned} (h^4 + \tau^2) \eta_i^\tau &= |b|(h^4 + \tau^2) \eta_{\dot{x}}^\tau + \frac{1}{2} |a|(h^4 + \tau^2) \eta_{\dot{x}\dot{x}}^\tau + q(h^4 + \tau^2) \eta_i^\tau \\ &+ (h^4 + h^2 \tau) \xi_{11} + (h^2 \tau + \tau^2) \xi_{10} \quad \text{on } Q_h^\tau. \end{aligned} \tag{3.38}$$

From the elementary inequality $(ab) \leq (1/2)(a^2 + b^2) < (a^2 + b^2)$, $a \geq 0, b \geq 0$, it follows that

$$h^2\tau \leq \frac{1}{2}(h^4 + \tau^2) < (h^4 + \tau^2).$$

Dividing both sides of Eq. (3.38) by $(h^4 + \tau^2)$ and using the last inequality we obtain:

$$\eta_i = |b|\eta_x^\tau + \frac{1}{2}|a|\eta_{xx}^\tau + q\eta^\tau + \xi_{12} \quad \text{on } Q_h^\tau, \tag{3.39}$$

where

$$|\xi_{12}| \leq 2 \sum_{i=1}^9 c_i = c_{10}.$$

To find estimate (3.29), inequality (3.27) is used after specifying the boundary and initial data for the net function η^τ . However, for $t = 0$ we have

$$\begin{aligned} \widehat{H}(x, 0) &= v(x, 0) = w(x, 0) = 0 \quad \forall x \in [0, \widehat{K}], \\ \widehat{H}^\tau(x, 0) &= 0 \quad \forall x \in \bar{\omega}_h. \end{aligned}$$

Therefore, it follows from Eq. (3.28) that

$$\eta^\tau(x, 0) = 0 \quad \forall x \in \bar{\omega}_h.$$

Similarly, using corresponding boundary conditions we check that

$$\eta^\tau(0, t) = \eta^\tau(\widehat{K}, t) = 0 \quad \forall t \in \omega_\tau.$$

We now apply estimate (3.27) to Eq. (3.39) to obtain:

$$\max_{\bar{Q}_h} |\eta^\tau| \leq T \max_{Q_h} |\xi_{12}| \leq c_{10}T. \tag{3.40}$$

Thus estimate (3.29) holds with constant $c_1 = c_{10}T$, which completes the proof of Theorem 3.1. \square

Secondly, the case $\Theta = \frac{1}{2}$ (Crank–Nicolson scheme) is considered in a similar way. Using the Crank–Nicolson scheme, Eq. (3.23) reduces to:

$$\begin{aligned} \widehat{H}_i^\tau\left(x, t + \frac{\tau}{2}\right) + b\left(x, t + \frac{\tau}{2}\right)\widehat{H}_x^\tau\left(x, t + \frac{\tau}{2}\right) + \frac{1}{2}a\left(x, t + \frac{\tau}{2}\right)\widehat{H}_{xx}^\tau\left(x, t + \frac{\tau}{2}\right) \\ = q\left(t + \frac{\tau}{2}\right)\widehat{H}^\tau\left(x, t + \frac{\tau}{2}\right) + f\left(x, t + \frac{\tau}{2}\right) \quad \text{on } Q_h^\tau. \end{aligned} \tag{3.41}$$

As in the case of the implicit method, the initial and boundary conditions have the form:

$$\widehat{H}^\tau(x, 0) = 0, \quad x \in \omega_h, \tag{3.42}$$

$$\widehat{H}^\tau(0, t) = \widehat{H}^\tau(\widehat{K}, t) = 0, \quad t \in \omega_\tau. \quad (3.43)$$

The following theorem is similar to Theorem 3.1, but, whereas Theorem 3.1 only offers second-order error on h , $O(h^2 + \tau)$, Theorem 3.2 offers second-order error in both h and τ , $O(h^2 + \tau^2)$.

Theorem 3.2 (See [6]). *Suppose that the conditions of Theorem 2.1, with $k = 4$, hold for (2.20)–(2.22). Then, the solution of the Crank–Nicolson difference scheme (3.41)–(3.43) can be written as:*

$$\widehat{H}^\tau = \widehat{H} + h^2 u + \tau^2 z + (h^4 + \tau^4) \eta^\tau \quad \text{on } \overline{Q}_h^\tau, \quad (3.44)$$

where the functions u and z are continuous on \overline{Q} and are independent of τ and h , and the net function η^τ is uniformly bounded:

$$|\eta^\tau| \leq c_1 \quad \text{on } \overline{Q}_h^\tau. \quad (3.45)$$

Proof. The proof of this theorem is almost identical to the proof of Theorem 3.1. We will only stress some of the minor changes in the proof and refer the reader to the proof of Theorem 3.1 if confusion arises. Also, the order of the proof here is a little different from that of the previous theorem. Substituting (3.44) into (3.41) we obtain:

$$\begin{aligned} & \widehat{H}_i + h^2 u_i + \tau^2 z_i + (h^4 + \tau^4) \eta_i^\tau + b \left[\widehat{H}_x + h^2 u_x + \tau^2 z_x + (h^4 + \tau^4) \eta_x^\tau \right] \\ & + \frac{1}{2} a \left[\widehat{H}_{xx} + h^2 u_{xx} + \tau^2 z_{xx} + (h^4 + \tau^4) \eta_{xx}^\tau \right] \\ & = q \left[\widehat{H} + h^2 u + \tau^2 z + (h^4 + \tau^4) \eta^\tau \right] + f \quad \text{on } Q_h^\tau. \end{aligned} \quad (3.46)$$

Using Lemmas A.1–A.3 from the Appendix A we have the following expansions:

$$\begin{aligned} \widehat{H}_i &= \frac{\partial \widehat{H}}{\partial t} + \frac{\tau^2}{24} \frac{\partial^3 \widehat{H}}{\partial t^3} + \tau^4 \xi_1, \\ u_i &= \frac{\partial u}{\partial t} + \tau^2 \xi_2, \\ z_i &= \frac{\partial z}{\partial t} + \tau^2 \xi_3, \\ \widehat{H}_{xx} &= \frac{\partial^2 \widehat{H}}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 \widehat{H}}{\partial x^4} + h^4 \xi_4, \\ u_{xx} &= \frac{\partial^2 u}{\partial x^2} + h^2 \xi_5, \\ z_{xx} &= \frac{\partial^2 z}{\partial x^2} + h^2 \xi_6, \end{aligned}$$

$$\begin{aligned} \widehat{H}_x &= \frac{\partial \widehat{H}}{\partial x} + \frac{h^2}{6} \frac{\partial^3 \widehat{H}}{\partial x^3} + h^4 \xi_7, \\ u_x &= \frac{\partial u}{\partial x} + h^2 \xi_8, \\ z_x &= \frac{\partial z}{\partial x} + h^2 \xi_9, \end{aligned}$$

where $|\xi_i| \leq c_i, i = 1, \dots, 9$ and the constants c_i are independent of the points of Q_h^τ and h, τ . Substituting the above expansions into (3.46) we obtain:

$$\begin{aligned} &\frac{\partial \widehat{H}}{\partial t} + \frac{1}{2} a \frac{\partial^2 \widehat{H}}{\partial x^2} + b \frac{\partial \widehat{H}}{\partial x} + h^2 \left(\frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} + \frac{1}{2} a \frac{\partial^2 u}{\partial x^2} - qu \right) \\ &+ \tau^2 \left(\frac{\partial z}{\partial t} + b \frac{\partial z}{\partial x} + \frac{1}{2} a \frac{\partial^2 z}{\partial x^2} - qz \right) + (h^2 \tau^2 + \tau^4) \xi_{10} + (h^4 + \tau^4) \eta_i^\tau \\ &+ (h^4 + h^2 \tau^2) \xi_{11} + b(h^4 + \tau^4) \eta_x^\tau + \frac{1}{2} a(h^4 + \tau^4) \eta_{xx}^\tau \\ &= h^2 \left(-\frac{1}{6} b \frac{\partial^3 \widehat{H}}{\partial x^3} - \frac{1}{24} a \frac{\partial^4 \widehat{H}}{\partial x^4} \right) + \tau^2 \left(-\frac{1}{24} \frac{\partial^3 \widehat{H}}{\partial t^3} \right) + q \widehat{H} \\ &+ q(h^4 + \tau^4) \eta^\tau + f \quad \text{on } Q_h^\tau. \end{aligned} \tag{3.47}$$

Here, $|\xi_{10}| \leq c_1 + c_2 + c_3$, and $|\xi_{11}| \leq |a|(c_4 + c_5 + c_6) + |b|(c_7 + c_8 + c_9)$. As in the proof of Theorem 3.1, the terms in h^0 and τ^0 cancel, since the function \widehat{H} satisfies Eq. (2.20). To cancel the terms of h^2 , we let the function u satisfy the following problem:

$$\frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} + \frac{1}{2} a \frac{\partial^2 u}{\partial x^2} - qu = h_1, \tag{3.48}$$

where

$$\begin{aligned} h_1 &= -\frac{1}{6} b \frac{\partial^3 \widehat{H}}{\partial x^3} - \frac{1}{24} a \frac{\partial^4 \widehat{H}}{\partial x^4}, \\ u(x, 0) &= 0, \quad x \in [0, \widehat{K}], \end{aligned} \tag{3.49}$$

$$u(0, t) = u(\widehat{K}, t) = 0, \quad t \in [0, T]. \tag{3.50}$$

By Theorem 2.1, the function $u \in H^4(\overline{Q})$. We can cancel the terms of τ^2 by letting z satisfy:

$$\frac{\partial z}{\partial t} + b \frac{\partial z}{\partial x} + \frac{1}{2} a \frac{\partial^2 z}{\partial x^2} - qz = h_2, \tag{3.51}$$

where

$$h_2 = -\frac{1}{24} \frac{\partial^3 \widehat{H}}{\partial t^3},$$

$$z(x, 0) = 0, \quad x \in [0, \widehat{K}], \quad (3.52)$$

$$z(0, t) = z(\widehat{K}, t) = 0, \quad t \in [0, T]. \quad (3.53)$$

By a similar reasoning as in the proof of Theorem 3.1, we have that $z \in H^{4,2}(\overline{Q})$. The rest of the proof is almost the same as in the proof of Theorem 3.1, and therefore, we arrive at:

$$\eta_i = |b|\eta_x^\tau + \frac{1}{2}|a|\eta_{xx}^\tau + q\eta^\tau + \xi_{12} \quad \text{on } Q_h^\tau, \quad (3.54)$$

where

$$|\xi_{12}| \leq 2 \sum_{i=1}^9 c_i = c_{10}.$$

Therefore

$$\max_{\overline{Q}_h} |\eta^\tau| \leq T \max_{Q_h^\tau} |\xi_{12}| \leq c_{10}T. \quad \square \quad (3.55)$$

4. Computational results

In this section, computational results are given for the three types of call options under consideration. We begin by examining European call options; then we consider American call options; and finally, we list the computational results for down-and-out knock-out call options.

In these calculational comparisons, the difference methods are applied directly to (2.2) with the appropriate initial and boundary conditions given by (2.3)–(2.8). Eqs. (2.20)–(2.22) are introduced in Section 2 to facilitate the analysis of the error expansions but, in practice, the difference schemes are applied directly to (2.2). Notice that to apply Richardson extrapolation to the Crank–Nicolson calculated values, τ and h must each change by a factor of 2 in mesh refinements, respectively. However, to correctly apply Richardson extrapolation to the calculated values obtained using the fully implicit difference scheme, τ and h must change by factors of 4 and 2, respectively, in mesh refinements. This is because by (3.28) the low order terms behave as $vh^2 + w\tau$, where v and w do not depend on h and τ . Then, the Richardson extrapolation approximations are calculated as described earlier. Specifically, $\hat{V}_{(h/2),(\tau/4)} = (4V_{(h/2),(\tau/4)} - V_{h,\tau})/3$, where $V_{h,\tau}$ is the approximate option value calculated using the fully implicit difference scheme with mesh widths of h and τ .

We list in Table 1 the results obtained for European call options with strike prices that are 95%, 100%, and 105% of the asset price. In this table, the Crank–Nicolson scheme is applied to two-year European call options with asset price of \$590. The computational results obtained by this method are

checked against the theoretical results obtained by the Black–Scholes formula. The interest rate is set to $r = 6\%$, and the dividend rate is set to $\gamma = 2.62\%$ (see, for comparison, Table 2, p. 27 in [1]). The results in the table indicate that Richardson extrapolation significantly increases the accuracy of the values obtained by the Crank–Nicolson scheme. Indeed the extrapolated values rapidly approach the exact values.

In Table 2 we list the computational results obtained for American call options with asset prices that are 90%, 100% and 110% of the strike price using the Crank–Nicolson scheme. This table is similar to the one above, except that we are considering American call options instead of European call options. In

Table 1
Two-year European call option prices: Crank–Nicolson scheme

K	σ	Number of intervals in S	Number of intervals in t	Approx. values	Extrap. values	Exact values
560.50	0.153	80	800	82.0414		83.5822
		160	1600	83.5346	84.0324	
		320	3200	83.5697	83.5813	
590.00	0.145	80	800	64.5539		64.8986
		160	1600	64.8132	64.8996	
		320	3200	64.8773	64.8987	
619.50	0.137	80	800	47.8994		48.2225
		160	1600	48.1352	48.2137	
		320	3200	48.2007	48.2226	

K = Strike price, σ = volatility, asset price = 590.

Table 2
Half-year American call option prices: Crank–Nicolson scheme

S	σ	Number of intervals in S	Number of intervals in t	Approx. values	Extrap. values	Exact values
90	0.4	32	192	5.616		5.722
		64	384	5.680	5.701	
		128	768	5.711	5.722	
		256	1536	5.718	5.721	
100	0.4	32	192	9.950		10.239
		64	384	10.167	10.239	
		128	768	10.220	10.238	
		256	1536	10.234	10.239	
110	0.4	32	192	18.452		16.181
		64	384	16.139	15.368	
		128	768	16.162	16.170	
		256	1536	16.176	16.181	

S = Asset price, σ = volatility, strike price = 100.

this table the length of time to maturity is six months. Also, all options have a strike price of $K = \$100$. The volatility is kept constant at $\sigma = 0.4$ for the three different computations. In addition, the interest rate is set to $r = 3\%$, while the dividend rate is set to $\gamma = 7\%$ (see Table 1, p. 1223 in [3] for comparison). From the table, we see that the computational values converge rapidly to the exact values even for a small mesh size. Indeed, when the asset price is $S = 100$, it is seen that the extrapolated values can reduce computational work by as much as a factor of 16.

In Table 3 we list the results obtained for down-and-out knock-out call options with strike prices being the same as the asset price, but with two different knock-out levels. In this table, both the asset price and the strike price are $S = K = \$590$. The interest rate and dividend rate are the same as in Table 1, but the volatilities are different (see Table 4, p. 29 in [1] for comparison).

Table 3
Two-year down-and-out knock-out call option prices: Crank–Nicolson scheme

Knock-out level	σ	Number of intervals in S	Number of intervals in t	Approx. values	Extrap. values	Exact values
520	0.12793	20	320	52.820		55.393
		40	640	54.810	55.473	
		80	1280	55.250	55.397	
		160	2560	55.358	55.393	
530	0.12416	20	320	50.530		52.279
		40	640	51.878	52.327	
		80	1280	52.180	52.281	
		160	2560	52.254	52.279	

σ = Volatility, asset price = 590, strike price = 590.

Table 4
Half-year American call option prices: implicit scheme

S	σ	Number of intervals in S	Number of intervals in t	Approx. values	Extrap. values	Exact values
90	0.4	32	384	5.609		5.722
		64	1536	5.675	5.698	
		128	6144	5.707	5.718	
100	0.4	32	384	9.941		10.239
		64	1536	10.162	10.235	
		128	6144	10.216	10.234	
110	0.4	32	384	18.444		16.181
		64	1536	16.134	15.364	
		128	6144	16.158	16.166	

S = Asset price, σ = volatility, strike price = 100.

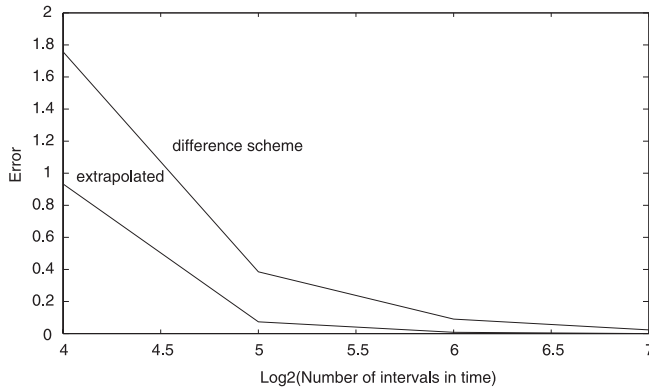


Fig. 1. Illustration of the error reduction possible by applying Richardson extrapolation to Crank–Nicolson option price estimates.

In Table 4 we list the results for American call options obtained using the implicit scheme. We have included this table to compare the two difference schemes under consideration. As expected, computational results indicate that the Crank–Nicolson scheme converges faster than the implicit scheme (compare these results with those of Table 2).

In Fig. 1 we show how rapidly the error of the extrapolated values decreases in relation to the error of the values obtained by Crank–Nicolson. In this figure, $S = K = \$100$, $r = 7\%$, $\gamma = 3\%$, $\sigma = 0.3$ and $T = 0.5$.

5. Conclusion

The Crank–Nicolson and fully implicit difference schemes were studied for calculation of American, European, and down-and-out knock-out call options. We proved that the error expansions for the difference methods have the correct form for applying Richardson extrapolation to increase the order of accuracy of the approximations. Computational results indicate that Richardson extrapolation can significantly decrease the amount of computational work in estimation of option prices.

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Appendix A

We begin this important section with a list of notation.

A.1. Net regions

$$\begin{aligned}\bar{\omega}_\tau &= \{t_j = j\tau; j = 0, 1, \dots, M\}, \\ \omega_\tau &= \{t_j = j\tau; j = 1, 2, \dots, M\}, \\ \bar{\omega}_h &= \{x_i = ih; i = 0, 1, \dots, N\}, \\ \omega_h &= \{x_i = ih; i = 1, 2, \dots, N - 1\}, \\ \bar{Q}_h^\tau &= \bar{\omega}_h \times \bar{\omega}_\tau, \quad Q_h^\tau = \omega_h \times \omega_\tau.\end{aligned}$$

A.2. Finite differences

$$\begin{aligned}u_x(x) &= \{u(x+h) - u(x)\}/h, \\ u_{\bar{x}}(x) &= \{u(x+h) - u(x-h)\}/2h, \\ u_{\bar{x}\bar{x}}(x) &= \{u(x-h) - 2u(x) + u(x+h)\}/h^2, \\ u_t(t) &= \{u(t+\tau) - u(t)\}/\tau, \\ u_{\bar{t}}(t) &= \{u(t+\tau/2) - u(t-\tau/2)\}/\tau, \\ u_{\bar{t}\bar{t}}(t) &= \{u(t) - u(t-\tau)\}/\tau.\end{aligned}$$

The following results are needed at several points in this paper. The following lemmas are simple consequences of Taylor's formula and are stated without proof. The reader is referred to [6] for a full exposition of their proofs.

Lemma A.1. *If the points $x, x \pm h$ belong to the interval $[0,1]$, then the function $v \in C^m[0, 1]$ has the following expansions:*

$$v_x(x) = \{v(x+h) - v(x)\}/h = \sum_{i=0}^{m-2} \frac{h^i}{(i+1)!} v^{(i+1)}(x) + h^{m-1} \zeta_1 \quad (\text{A.1})$$

and

$$v_{\bar{x}}(x) = \{v(x+h) - v(x-h)\}/2h = \sum_{i=0}^r \frac{h^{2i}}{(2i+1)!} v^{(2i+1)}(x) + h^{m-1} \zeta_2, \quad (\text{A.2})$$

where $r = [m/2] - 1$. All the ζ_i are bounded independent of x and h by

$$\frac{1}{m!} \max_{[0,1]} |v^{(m)}|.$$

Lemma A.2. *If the points $t, t \pm \tau$ belong to the interval $[0,1]$, then the function $v \in C^n[0, 1]$ has the following expansions*

$$v_i(t) = \{v(t + \tau) - v(t)\}/\tau = \sum_{j=0}^{n-2} \frac{\tau^j}{(j + 1)!} v^{(j+1)}(t) + \tau^{n-1} \kappa_1, \tag{A.3}$$

$$\begin{aligned} v_i(t) &= \{v(t + \tau/2) - v(t - \tau/2)\}/\tau \\ &= \sum_{j=0}^s \frac{\tau^{2j}}{(2j + 1)! 4^j} v^{(2j+1)}(t) + 2^{-n+1} \tau^{n-1} \kappa_2, \end{aligned} \tag{A.4}$$

where $s = [n/2] - 1$ and

$$v_i(t) = \{v(t) - v(t - \tau)\}/\tau = \sum_{j=0}^{n-2} \frac{(-\tau)^j}{(j + 1)!} v^{(j+1)}(t) + \tau^{n-1} \kappa_3. \tag{A.5}$$

All the κ_k are bounded independent of t and τ by

$$\frac{1}{n!} \max_{[0,1]} |v^{(n)}|.$$

Lemma A.3. *Let $v \in C^{l+\alpha}[x - h, x + h]$ with integer $l \geq 2$, and $\alpha \in (0, 1)$. Then we have*

$$\begin{aligned} v_{xx}(x) &= \{v(x - h) - 2v(x) + v(x + h)\}/h^2 \\ &= 2 \sum_{i=0}^s \frac{h^{2i}}{(2i + 2)!} v^{(2i+1)}(x) + h^{l-2+\alpha} \rho(x), \end{aligned} \tag{A.6}$$

where $s = [l/2] - 1$ and

$$|\rho(x)| \leq \frac{2}{l!} \|v\|_{C^{l+\alpha}[x-h, x+h]}.$$

Appendix B. A smooth initial condition

In this section, it is explained how problem (2.10)–(2.12) with a non-smooth initial condition can be approximated by problem (2.15) and (2.16) with a smooth initial condition. We consider the main problem:

$$\frac{\partial H(x, t)}{\partial t} + \frac{1}{2} \hat{a}(x, t) \frac{\partial^2 H(x, t)}{\partial x^2} + \hat{b}(x, t) \frac{\partial H(x, t)}{\partial x} = q(t)H(x, t), \tag{B.7}$$

$$H(0, t) = H(\hat{K}, t) = 0. \tag{B.8}$$

Letting $K_1 = \widehat{K}/2$ and $K_2 = \widehat{K} - K$, we have

$$h(x) \equiv H(x, 0) = \begin{cases} 0 & \text{if } 0 \leq x \leq K, \\ x - K & \text{if } K \leq x \leq K_1, \\ \frac{(K_2 - x)(K_1 - x)}{K_2 - K_1} & \text{if } K_1 \leq x \leq K_2, \\ 0 & \text{if } K_2 \leq x \leq \widehat{K}, \end{cases}$$

where \widehat{K} is sufficiently large, i.e., $\widehat{K} > 2K$. Letting $z = -1 + (2x/\widehat{K})$ and $x = (z + 1)(\widehat{K}/2)$, then

$$\frac{\partial H(x, t)}{\partial t} + \frac{1}{2}a(x, t) \frac{\partial^2 H(x, t)}{\partial x^2} + b(x, t) \frac{\partial H(x, t)}{\partial x} = q(t)H(x, t), \tag{B.9}$$

$$H(-1, t) = H(1, t) = 0. \tag{B.10}$$

Therefore,

$$h(z) \equiv H(z, 0) = \begin{cases} 0 & \text{if } -1 \leq z \leq \frac{2K}{\widehat{K}} - 1, \\ (z + 1) \frac{\widehat{K}}{2} - K & \text{if } \frac{2K}{\widehat{K}} - 1 \leq z \leq 0, \\ \widehat{K} - K - (z + 1) \frac{\widehat{K}}{2} & \text{if } 0 \leq z \leq 1 - \frac{2K}{\widehat{K}}, \\ 0 & \text{if } 1 - \frac{2K}{\widehat{K}} \leq z \leq 1. \end{cases}$$

Letting $d = 1 - (2K/\widehat{K})$ and $r = (\widehat{K}/2)$, where $\widehat{K} > 2K$, $0 < d < 1$, we define $\widehat{H}(x, t)$ to be the solution of

$$\frac{\partial \widehat{H}(x, t)}{\partial t} + \frac{1}{2}a(x, t) \frac{\partial^2 \widehat{H}(x, t)}{\partial x^2} + b(x, t) \frac{\partial \widehat{H}(x, t)}{\partial x} = q(t)\widehat{H}(x, t), \tag{B.11}$$

$$\widehat{H}(0, t) = \widehat{H}(\widehat{K}, t) = 0, \tag{B.12}$$

so that

$$\widehat{H}(z, 0) = \hat{h}(z) \approx h(z) = \begin{cases} 0 & \text{if } -1 \leq z \leq -d, \\ rz + rd & \text{if } -d \leq z \leq 0, \\ -rz + rd & \text{if } 0 \leq z \leq d, \\ 0 & \text{if } d \leq z \leq 1, \end{cases}$$

where $\hat{h}(z)$ is a C^∞ function. To define $\hat{h}(z)$, notice that

$$h(z) = b_0 + \sum_{n=1}^{\infty} b_n \cos(\pi n z) = \begin{cases} 0 & \text{if } -1 \leq z \leq -d, \\ rz + rd & \text{if } -d \leq z \leq 0, \\ -rz + rd & \text{if } 0 \leq z \leq d, \\ 0 & \text{if } d \leq z \leq 1, \end{cases}$$

where

$$b_0 = \int_0^1 h(z) dz = \int_0^d (-rz + rd) dz = \frac{rd^2}{2}$$

and

$$\begin{aligned} b_n &= 2 \int_0^1 h(z) \cos(\pi n z) dz = 2 \int_0^d (-rz + rd) \cos(\pi n z) dz \\ &= -2r \frac{\cos(\pi n d) - 1}{(\pi n)^2}. \end{aligned}$$

Thus

$$h(z) = \frac{rd^2}{2} + \sum_{n=1}^{\infty} -2r \frac{\cos(\pi n d) - 1}{(\pi n)^2} \cos(\pi n z).$$

Finally, $\hat{h}(z)$ is defined as

$$\hat{h}(z) = \frac{rd^2}{2} + \sum_{n=1}^{\infty} -2r \frac{\cos(\pi n d) - 1}{(\pi n)^2} \cos(\pi n z) \exp^{-(\pi n)^2 \hat{a}},$$

where \hat{a} is a small positive number. Notice that $\hat{h} \in C^\infty[-1, 1]$, and it is straightforward to show that $\|h - \hat{h}\|_2$ and $\|h' - \hat{h}'\|_2$ can be made arbitrarily small by making the value of \hat{a} sufficiently small.

Now, let $H^*(z, t) = H(z, t) - \hat{H}(z, t)$ and notice that H^* satisfies (B.11) and (B.12) with the initial condition $H^*(z, 0) = h(z) - \hat{h}(z)$. Assuming that $a, b \in H^{2,1}(\bar{Q})$, it follows from Theorem 9.1 (p. 341) of [4] that there is a constant $c > 0$ such that

$$\|H^*\|_2 \leq c(\|h - \hat{h}\|_2 + \|h' - \hat{h}'\|_2).$$

This inequality implies that by making the value of \hat{a} sufficiently small, \hat{H} satisfying (B.11) and (B.12) with a smooth initial condition can be made arbitrarily close to H , which satisfies (B.9) and (B.10).

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