Stochastic Analysis and Applications, 25: 187–200, 2007

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362990601052227

Shooting Methods for Numerical Solution of Nonlinear Stochastic Boundary-Value Problems

Armando Arciniega
Department of Mathematics, The University of Texas,
San Antonio, Texas, USA

Abstract: In the present investigation, shooting methods are described for numerically solving nonlinear stochastic boundary-value problems. These stochastic shooting methods are analogous to standard shooting methods for numerical solution of ordinary deterministic boundary-value problems. It is shown that the shooting methods provide accurate approximations. An error analysis is performed and computational simulations are described.

Keywords: Broyden's method; Itô and Stratonovich stochastic differential equations; Nonlinear stochastic boundary-value problems; Shooting methods.

Mathematics Subject Classification: 60H35; 65C30; 60H10.

1. INTRODUCTION

Methods for numerically solving stochastic initial-value problems have been under much study (see, e.g., [2, 5, 7, 8, 15, 16] and the references therein.) However, the theory and numerical solution of stochastic boundary-value problems have received less attention [12, 17]. Generally, these stochastic boundary-value problems cannot be solved exactly and numerical methods must be used to obtain an approximate solution.

Received July 13, 2006; Accepted September 8, 2006
The author is grateful to Edward J. Allen for many helpful discussions.
Address correspondence to Armando Arciniega, Department of Mathematics, The University of Texas at San Antonio, One UTSA Circle, San Antonio, TX 78249, USA; E-mail: armando.arciniega@utsa.edu

In recent work, Arciniega and Allen [3] applied a shooting method procedure to numerically solve linear systems of Stratonovich boundary-value problems. In this work, it was proved that the shooting method procedure accurately approximates the solutions of linear stochastic boundary-value problems, an error analysis of the shooting method procedure was performed, and computational simulations were described.

The purpose of the present investigation is to extend the previous work on shooting methods in Arciniega and Allen [3] for numerical solution of nonlinear stochastic boundary-value problems. These stochastic shooting methods are similar to standard shooting methods for numerical solution of ordinary deterministic boundary-value problems. An error analysis of these methods is performed and computational simulations are described. The analysis of this nonlinear method is similar to that for the linear problem reported in Arciniega and Allen [3]. However, the method described in this investigation is more general and is applicable to both linear and nonlinear stochastic boundary-value problems. In addition, this method may be useful in certain applications. For example, this method can be applied to the stochastic neutron transport equations derived in Sharp and Allen [14].

In the next section, the shooting method procedure for solution of a nonlinear stochastic boundary-value problem is described. In the third section, Broyden's method is employed in the shooting method procedure for iterative solution. An error analysis is performed in the fourth section. Finally, computational results are presented to illustrate the numerical procedure.

2. DESCRIPTION OF THE SHOOTING METHOD PROCEDURE

Consider the following multidimensional, nonlinear stochastic boundary-value problem:

$$\begin{cases} d\vec{X}(t) + \vec{f}(t, \vec{X}(t))dt = Md\vec{W}(t), \quad 0 \leq t \leq 1 \\ F_0\vec{X}(0) + F_1\vec{X}(1) = \vec{\beta}, \end{cases} \tag{2.1}$$

where $\vec{X}$ takes values in $\mathbb{R}^n$, $\vec{f} : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$, M is a $n \times k$ matrix, $\vec{\beta} \in \mathbb{R}^n$, and $\{\vec{W}(t), t \in [0, 1]\}$ is a standard k-dimensional Wiener process. In addition, F_0 and F_1 are $n \times n$ matrices such that $\text{rank}(F_0 : F_1) = n$. It is assumed in the present investigation that the matrix M is independent of t. The stochastic integrals for this problem can be interpreted as Itô or Stratonovich as they are identical for M a constant matrix (see, e.g., Kloeden and Platen [7]). Existence and uniqueness of a solution, under smoothness and monotonicity conditions on $\vec{f}$, have been proved by Nualart and Pardoux [10, 11]. As described in Nualart and Pardoux [10], a solution of (2.1), $\vec{X} \in C([0, 1]; \mathbb{R}^n)$ satisfies the equivalent stochastic integral equation:

$$\begin{cases} \vec{X}(t) + \int_0^t \vec{f}(s, \vec{X}(s))ds = \vec{X}(0) + M\vec{W}(t), & 0 \le t \le 1 \\ F_0\vec{X}(0) + F_1\vec{X}(1) = \vec{\beta}. \end{cases} \tag{2.2}$$

To motivate the shooting method procedure for solution of (2.1), consider the scalar second-order, nonlinear two-point stochastic boundary-value problem with Dirichlet boundary conditions [11]:

$$\begin{cases} \dfrac{d^2X(t)}{dt^2} + f\left(X(t), \dfrac{dX(t)}{dt}\right) = \dfrac{dW(t)}{dt}, & 0 \le t \le 1 \\ X(0) = a \\ X(1) = b. \end{cases} \tag{2.3}$$

By a change of variables, (2.3) can be written as a system of two first-order equations:

$$\begin{cases} dX_1(t) = X_2(t)dt \\ dX_2(t) = -f(X_1(t), X_2(t))dt + dW(t), & 0 \le t \le 1 \\ X_1(0) = a \\ X_1(1) = b. \end{cases} \tag{2.4}$$

Notice that (2.4) is of the form (2.1):

$$d\vec{X}(t) = \begin{pmatrix} X_2(t)dt \\ -f(t, \vec{X}(t)) \end{pmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dW_1(t), \quad 0 \le t \le 1$$

where $X_1(t) = X(t)$, $X_2(t) = X'(t)$, and $\vec{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$ with

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \vec{X}(0) = \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix},$$

$$\vec{X}(1) = \begin{bmatrix} X_1(1) \\ X_2(1) \end{bmatrix}, \quad \text{and} \quad \vec{\beta} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

In the shooting method procedure, the stochastic boundary-value problem (2.4) is replaced with the corresponding stochastic initial-value problem (see, e.g., Keller [6]):

$$\begin{cases} dU_1(t) = U_2(t)dt \\ dU_2(t) = -f(U_1(t), U_2(t))dt + dW(t), & 0 \le t \le 1 \\ U_1(0) = a \\ U_2(0) = \alpha. \end{cases} \tag{2.5}$$

The stochastic initial-value problem (2.5) has a unique solution under smoothness and monotonicity conditions on f. Notice that (2.5) just another way of writing the scalar second-order, nonlinear stochastic initial-value problem:

$$\begin{cases} \dfrac{d^2U(t)}{dt^2} + f\left(U(t), \dfrac{dU(t)}{dt}\right) = \dfrac{dW(t)}{dt}, \quad 0 \le t \le 1 \\ U(0) = a \\ \frac{dU}{dt}(0) = \alpha, \end{cases} \tag{2.6}$$

which corresponds to the scalar second-order, nonlinear two-point stochastic boundary-value (2.3). Let $U(t; \alpha)$ be the unique solution of the initial-value problem (2.5)–(2.6) and define $R(\alpha) \equiv U(1; \alpha) - b$. If α^* is a root of the nonlinear function $R(\alpha^*) = 0$, then it is clear that $X^*(t) \equiv U(t; \alpha^*)$ is a solution of the boundary-value problem (2.3)–(2.4). In other words, the purpose of solving initial-value problem (2.5)–(2.6) is to find a value of α that gives a solution satisfying $U(1) = b$ of the corresponding boundary-value problem (2.3)–(2.4). That is, the objective is to find a root of the nonlinear function $R(\alpha) \equiv U(1; \alpha) - b = 0$, where $U(t; \alpha)$ represents the unique solution of the stochastic initial-value problem (2.5)–(2.6). Notice that $U(t) \equiv U(t; \alpha)$ explicitly indicates the dependence of the solution on α as well as on t. Figure 1 illustrates the shooting method procedure for problem (2.5)–(2.6).

Notice that an explicit solution of initial-value problem (2.5)–(2.6) is generally not obtainable, and one must use a numerical procedure to find an approximate solution. In practice, stochastic initial-value problem (2.5)–(2.6) is numerically solved using standard numerical methods, such as the Euler-Maruyama method or Milstein's method. In addition, the value of α must be estimated. An appropriate method for finding an

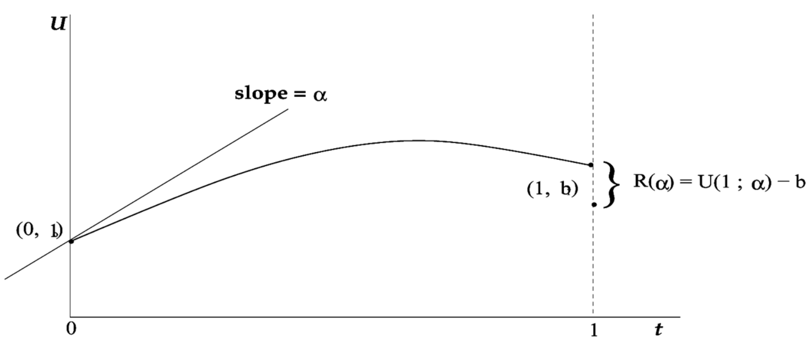

Figure 1. Illustration of the shooting method procedure for problem (2.5)–(2.6).

approximate root of $R(\alpha) = 0$ is the secant method:

$$\alpha_{i+1} = \alpha_i - \frac{R(\alpha_i)(\alpha_i - \alpha_{i-1})}{R(\alpha_i) - R(\alpha_{i-1})},$$

which converges rapidly near a root and requires no derivatives of $R(\alpha)$. Therefore, the shooting method procedure consists of an iterative scheme for approximating the root of the nonlinear function $R(\alpha) \equiv U(1;\alpha) - b = 0$, where $U(1;\alpha)$ is obtained through numerical solution of stochastic initial-value problem (2.5)–(2.6).

Consider briefly the special case when the stochastic differential equation in (2.5)–(2.6) is linear. Let $Z(t)$ be the solution of the corresponding homogeneus stochastic differential equation satisfying $Z(0) = 0$ and $Z'(0) = 1$. Then $R(\alpha)$ is a linear function of α, as $U(t;\alpha) = U(t;0) + \alpha Z(t)$ and thus $R(\alpha) = U(1;0) + \alpha Z(1) - b$, where $U(t;0)$ is the solution of (2.5)–(2.6) with $\alpha = 0$. Therefore, only two initial-value computations, with $\alpha = 0$ and $\alpha = 1$ are necessary to compute the correct value of α and the solution. Indeed, $R(\alpha) = 0$ yields

$$\alpha = \frac{b - U(1;0)}{Z(1)}.$$

Work in the linear case was investigated in Arciniega and Allen [3]. The shooting method described in this investigation is more general and is applicable to both linear and nonlinear stochastic boundary-value problems.

The shooting method procedure for solution of the multidimensional, nonlinear stochastic boundary-value problem:

$$\begin{cases} d\vec{X}(t) + \vec{f}(t, \vec{X}(t))dt = Md\vec{W}(t), \quad 0 \le t \le 1 \\ F_0\vec{X}(0) + F_1\vec{X}(1) = \vec{\beta}, \end{cases} \tag{2.7}$$

is similar to the procedure in the scalar case. One determines a vector $\vec{\alpha} \in \mathbb{R}^n$ for the corresponding stochastic initial-value problem:

$$\begin{cases} d\vec{U}(t) = -\vec{f}(t, \vec{U}(t))dt + Md\vec{W}(t), \quad 0 \le t \le 1 \\ \vec{U}(0) = \vec{\alpha} \end{cases} \tag{2.8}$$

in such a way that the solution $\vec{U}(t) \equiv \vec{U}(t;\vec{\alpha})$ obeys the boundary conditions in (2.7):

$$F_0\vec{U}(0;\vec{\alpha}) + F_1\vec{U}(1;\vec{\alpha}) = F_0\vec{\alpha} + F_1\vec{U}(1;\vec{\alpha}) = \vec{\beta}.$$

That is, one has to find a root

$$\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$$

of a nonlinear system

$$\vec{R}(\vec{\alpha}) = \vec{0}, \quad \vec{R}(\vec{\alpha}) \equiv F_0\vec{U}(0;\vec{\alpha}) + F_1\vec{U}(1;\vec{\alpha}) - \vec{\beta} \tag{2.9}$$

such that (2.7) is satisfied. To see this, let $\vec{U}(t;\vec{\alpha})$ be the unique solution of initial-value problem (2.8). If $\vec{\alpha}^*$ is a root of the nonlinear function $R(\vec{\alpha}^*) = \vec{0}$, then $\vec{X}^*(t) \equiv \vec{U}(t;\vec{\alpha}^*)$ is a solution of the boundary-value problem (2.7). However, explicit solution of (2.9) is generally not obtainable, and one must use an iterative procedure to find the solution. Iterative solution of (2.9), using Broyden's method is discussed in the next section.

3. NUMERICAL SOLUTION AND BROYDEN'S METHOD

In the previous section, the shooting method procedure was described for solution of nonlinear stochastic boundary-value problems. In this section, the shooting method procedure for numerical solution of

$$\begin{cases} d\vec{X}(t) + \vec{f}(t,\vec{X}(t))dt = Md\vec{W}(t), & 0 \le t \le 1 \\ F_0\vec{X}(0) + F_1\vec{X}(1) = \vec{\beta}, \end{cases} \tag{3.1}$$

is described. An error analysis is performed in the next section. To obtain an approximate solution, stochastic boundary-value problem (3.1) is replaced with a corresponding stochastic initial-value problem:

$$\begin{cases} d\vec{U}(t) = -\vec{f}(t,\vec{U}(t))dt + Md\vec{W}(t), & 0 \le t \le 1 \\ \vec{U}(0) = \vec{\alpha} \end{cases} \tag{3.2}$$

such that $F_0\vec{U}(0;\vec{\alpha}) + F_1\vec{U}(1;\vec{\alpha}) = \vec{\beta}$ is satisfied. That is, one has to find a root $\vec{\alpha} \in \mathbb{R}^n$ of a nonlinear system:

$$\vec{R}(\vec{\alpha}) \equiv F_0\vec{U}(0;\vec{\alpha}) + F_1\vec{U}(1;\vec{\alpha}) - \vec{\beta} = F_0\vec{\alpha} + F_1\vec{U}(1;\vec{\alpha}) - \vec{\beta} = \vec{0}. \tag{3.3}$$

The numerical method for solving (3.1) consists of the following steps. An initial $\vec{\alpha}_0$ is guessed. One possible choice of $\vec{\alpha}_0$ may be the vector that solves (3.1)–(3.2) with $M = 0$ which, of course, must be computationally approximated beforehand. Once $\vec{\alpha}_0$ is available, the Jacobian matrix $\vec{R}'(\vec{\alpha}_0)$ is computed. To update $\vec{R}(\vec{\alpha}_0)$ and find $\vec{\alpha}_1$ the quasi-Newton Broyden's method is employed. For each $\vec{\alpha}_p$, (3.2) is solved numerically to calculate $\vec{U}(1;\vec{\alpha}_p)$ in order to find $\vec{R}(\vec{\alpha}_p) = F_0\vec{U}(0;\vec{\alpha}_p) + F_1\vec{U}(1;\vec{\alpha}_p) - \vec{\beta}$. Thus, for each iteration of Broyden's method, initial-value problem (3.2) is numerically solved in order to compute $\vec{U}(1;\vec{\alpha}_p)$ and hence $\vec{R}(\vec{\alpha}_p)$. In summary, Broyden's method is used to numerically approximate the root of the nonlinear system (3.3), where numerical solution of stochastic initial-value problem (3.2) is performed at each step of Broyden's method.

Notice that to solve initial-value problem (3.2) numerically, the interval [0, 1] is partitioned into N subintervals of equal length $h = \frac{1}{N}$ with $t_p = ph$ for each $p = 0, 1, \ldots, N$. The Broyden approximations of (3.3) satisfy the iterative scheme (see, e.g. Keller [6], and Ortega [13]):

$$\begin{cases} \vec{\alpha}_{p+1} = \vec{\alpha}_p - B_p^{-1}\vec{R}(\vec{\alpha}_p), \\ B_{p+1} = B_p + \dfrac{(\vec{y}_p - B_p\vec{s}_p)\vec{s}_p^T}{\vec{s}_p^T\vec{s}_p} \\ \vec{y}_p = \vec{R}(\vec{\alpha}_{p+1}) - \vec{R}(\vec{\alpha}_p) \\ \vec{s}_p = \vec{\alpha}_{p+1} - \vec{\alpha}_p \end{cases} \tag{3.4}$$

where $\vec{\alpha}_p$ denotes the approximation to the root of (3.3). That is $\vec{\alpha}_p \approx \vec{\alpha}$, for each $p = 0, 1, \ldots$. Notice that Broyden's method requires the inverse of a nonsingular matrix B for each iteration. The matrix B is an approximation of the Jacobian of the vector-valued function $\vec{R}(\vec{\alpha})$.

4. ERROR ANALYSIS

In this section, an error analysis is performed. In this analysis, it is assumed that the Broyden iterations have continued until convergence has occurred. The error in the approximate solution is caused by the error in approximating the initial-value problem (3.2). Recall that we are trying to solve the nonlinear stochastic boundary-value problem:

$$\begin{cases} d\vec{X}(t) + \vec{f}(t, \vec{X}(t))dt = Md\vec{W}(t), \quad 0 \le t \le 1 \\ F_0\vec{X}(0) + F_1\vec{X}(1) = \vec{\beta}. \end{cases} \tag{4.1}$$

Problem (4.1) is replaced with the corresponding stochastic initial-value problem:

$$\begin{cases} d\vec{U}(t) = -\vec{f}(t, \vec{U}(t))dt + Md\vec{W}(t), \quad 0 \le t \le 1 \\ \vec{U}(0) = \vec{\alpha} \end{cases} \tag{4.2}$$

with the objective of finding a vector $\vec{\alpha} \in \mathbb{R}^n$ such that

$$F_0\vec{U}(0; \vec{\alpha}) + F_1\vec{U}(1; \vec{\alpha}) = F_0\vec{\alpha} + F_1\vec{U}(1; \vec{\alpha}) = \vec{\beta} \tag{4.3}$$

is satisfied.

To perform an error analysis for numerical solution of (4.2) satisfying (4.3), the following notation is introduced. Let $\vec{U}(t;\vec{\gamma})$ be the solution of initial-value problem (4.2), $L\vec{U}=\vec{0}$ with initial condition $\vec{U}(0;\vec{\gamma})=\vec{\gamma}$. Next, let $\vec{U}_h(t;\vec{\gamma})$ be the solution of an approximation to (4.2) using a step length of h. That is, $\vec{U}_h(t;\vec{\gamma})$ satisfies $L_h\vec{U}_h=\vec{0}$ with $\vec{U}_h(0;\vec{\gamma})=\vec{\gamma}$.

Using the notation in the preceeding paragraph, let $\vec{U}(t;\vec{\alpha})$ satisfy $L\vec{U}(t;\vec{\alpha})=\vec{0}$ with $\vec{U}(0;\vec{\alpha})=\vec{\alpha}$ and $F_0\vec{\alpha}+F_1\vec{U}(1;\vec{\alpha})=\vec{\beta}$. Next, let $\vec{U}_h(t;\vec{\alpha}^\star)$ satisfy $L_h\vec{U}_h(t;\vec{\alpha}^\star)=\vec{0}$ with $\vec{U}_h(0;\vec{\alpha}^\star)=\vec{\alpha}^\star$ and $F_0\vec{\alpha}^\star+F_1\vec{U}_h(1;\vec{\alpha}^\star)=\vec{\beta}$. Notice that $\vec{\alpha}$ is not exactly calculated, as a numerical scheme approximates the initial-value problem (4.2). The approximation to $\vec{\alpha}$ is denoted as $\vec{\alpha}^\star$. We need to estimate (see, e.g., Kloeden and Platen [7] and Milstein and Tretyakov [9]):

$$\|\vec{U}(t;\vec{\alpha})-\vec{U}_h(t;\vec{\alpha}^\star)\|^2=E\left(\sum_{i=1}^{n}(U_i(t;\vec{\alpha})-U_{h,i}(t;\vec{\alpha}^\star))^2\right).$$

Consider

$$\begin{cases}F_0\vec{\alpha}+F_1\vec{U}(1;\vec{\alpha})=\vec{\beta}\\ F_0\vec{\alpha}^\star+F_1\vec{U}_h(1;\vec{\alpha}^\star)=\vec{\beta}.\end{cases}\tag{4.4}$$

By Taylor series,

$$\begin{aligned}\vec{U}(1;\vec{\alpha})&=\vec{U}(1;\vec{\alpha}^\star)+\vec{U}'(1;\vec{\alpha}^\star)(\vec{\alpha}-\vec{\alpha}^\star)+\mathcal{O}(\|\vec{\alpha}-\vec{\alpha}^\star\|^2)\\ &=\vec{U}(1;\vec{\alpha}^\star)+\vec{U}'(1;\vec{\alpha})(\vec{\alpha}-\vec{\alpha}^\star)+\mathcal{O}(\|\vec{\alpha}-\vec{\alpha}^\star\|^2)\end{aligned}$$

where $\vec{U}'(1;\vec{\alpha})$ is a Jacobian matrix. Then, (4.4) becomes

$$\begin{cases}F_0\vec{\alpha}+F_1\vec{U}(1;\vec{\alpha}^\star)+F_1\vec{U}'(1;\vec{\alpha})(\vec{\alpha}-\vec{\alpha}^\star)=\vec{\beta}+\mathcal{O}(\|\vec{\alpha}-\vec{\alpha}^\star\|^2)\\ F_0\vec{\alpha}^\star+F_1\vec{U}_h(1;\vec{\alpha}^\star)=\vec{\beta}.\end{cases}\tag{4.5}$$

Subtracting corresponding terms in (4.5) yields

$$F_0(\vec{\alpha}-\vec{\alpha}^\star)+F_1(\vec{U}(1;\vec{\alpha}^\star)-\vec{U}_h(1;\vec{\alpha}^\star))+F_1\vec{U}'(1;\vec{\alpha})(\vec{\alpha}-\vec{\alpha}^\star)=\mathcal{O}(\|\vec{\alpha}-\vec{\alpha}^\star\|^2).$$

Therefore,

$$(F_0+F_1\vec{U}'(1;\vec{\alpha}))(\vec{\alpha}-\vec{\alpha}^\star)=-F_1(\vec{U}(1;\vec{\alpha}^\star)-\vec{U}_h(1;\vec{\alpha}^\star))+\mathcal{O}(\|\vec{\alpha}-\vec{\alpha}^\star\|^2).$$

This implies

$$\|\vec{\alpha}-\vec{\alpha}^\star\|\le\|G^{-1}F_1\|_2\|\vec{U}(1;\vec{\alpha}^\star)-\vec{U}_h(1;\vec{\alpha}^\star)\|+\mathcal{O}(\|\vec{\alpha}-\vec{\alpha}^\star\|^2),$$

where $G = F_0 + F_1 \vec{U}'(1; \vec{\alpha})$ is assumed to be nonsingular and $\|\cdot\|_2$ is an induced matrix norm. It follows that $\|\vec{\alpha} - \vec{\alpha}^\star\| \leq c_1 h^p$, where $c_1 h^p$ is a bound on the error due to the numerical method for solving initial-value problem (4.2).

Now, consider

$$\begin{aligned}\|\vec{U}(t; \vec{\alpha}) - \vec{U}_h(t; \vec{\alpha}^\star)\| &\leq \|\vec{U}(t; \vec{\alpha}^\star) - \vec{U}_h(t; \vec{\alpha}^\star)\| + \|\vec{U}(t; \vec{\alpha}) - \vec{U}(t; \vec{\alpha}^\star)\| \\ &\leq c_2 h^p + c_3 \|\vec{\alpha} - \vec{\alpha}^\star\|,\end{aligned}$$

where $c_2 h^p$ is a bound on the error due to the numerical method for solving initial-value problem (4.2) and $c_3 \|\vec{\alpha} - \vec{\alpha}^\star\|$ is a bound on the error due to the difference between having initial condition $\vec{\alpha}$ and initial condition $\vec{\alpha}^\star$ (see, e.g., Kloeden and Platen [7], and Milstein and Tretyakov [9]). For example, $p = 1/2$ for the Euler-Maruyama method. Hence,

$$\|\vec{U}(t; \vec{\alpha}) - \vec{U}_h(t; \vec{\alpha}^\star)\| \leq c_2 h^p + c_3 c_1 h^p = \mathcal{O}(h^p).$$

5. COMPUTATIONAL RESULTS

In this section, computational results are presented to test the numerical methods described in the present investigation. As a first example, consider the following scalar, second-order, nonlinear two-point stochastic boundary-value problem with Dirichlet boundary conditions:

$$\begin{cases} \dfrac{d^2X(t)}{dt^2} + f\left(X(t), \dfrac{dX(t)}{dt}\right) = M\dfrac{dW(t)}{dt}, & 0 \leq t \leq 1 \\ X(0) = 0 \\ X(1) = 0, \end{cases} \tag{5.1}$$

where

$$f\left(X(t), \frac{dX(t)}{dt}\right) = -\frac{1}{4}\cos X(t)$$

and

$$M = \frac{1}{10}.$$

This problem satisfies the smoothness and monotonicity conditions on f [10, 11]. Therefore, it has a unique solution. Notice that (5.1) can be

Table 1. Approximate values of $E(X_1(1/2))$ and $E(X_1^2(1/2))$

Number of Intervals in t	$E(X_1(1/2))$	$E(X_1^2(1/2))$
2	−0.031468	0.001305
4	−0.031294	0.001217
8	−0.031161	0.001189
16	−0.031340	0.001198
32	−0.031117	0.001176
64	−0.031047	0.001172

written as a system of two first-order equations, and has the form (2.1) with

$$\vec{f}(t, \vec{X}(t)) = \begin{bmatrix} -X_2(t) \\ -\frac{1}{4}\cos X_1(t) \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ .1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \vec{\beta} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Also, notice that for this problem $\{W(t), t \in [0, 1]\}$ is a 1-dimensional Wiener process. Table 1 presents approximations of $E(X_1(1/2))$ and $E(X_1^2(1/2))$ using the shooting method with Euler's method solving the stochastic initial-value systems and Broyden's method iteration used to update the $\vec{\alpha}$'s. The approximate values are based on 10,000 independent trials.

Figure 2 illustrates the average of the approximate solution with 10,000 independent trials using the shooting method procedure. A single trajectory of the numerical solution is also shown. The approximate derivative of the solution is also shown.

For a second example, consider the scalar, second-order, *but, linear* two-point stochastic boundary-value problem:

$$\begin{cases} X''(t) = (-1 + X(t))dt + \frac{1}{10}dW(t) \\ X(0) = 0 \\ X(1) = 0. \end{cases} \tag{5.2}$$

In the form (2.1), one has

$$\vec{f}(t, \vec{X}(t)) = \begin{bmatrix} -X_2(t) \\ 1 - X_1(t) \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ .1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{\beta} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

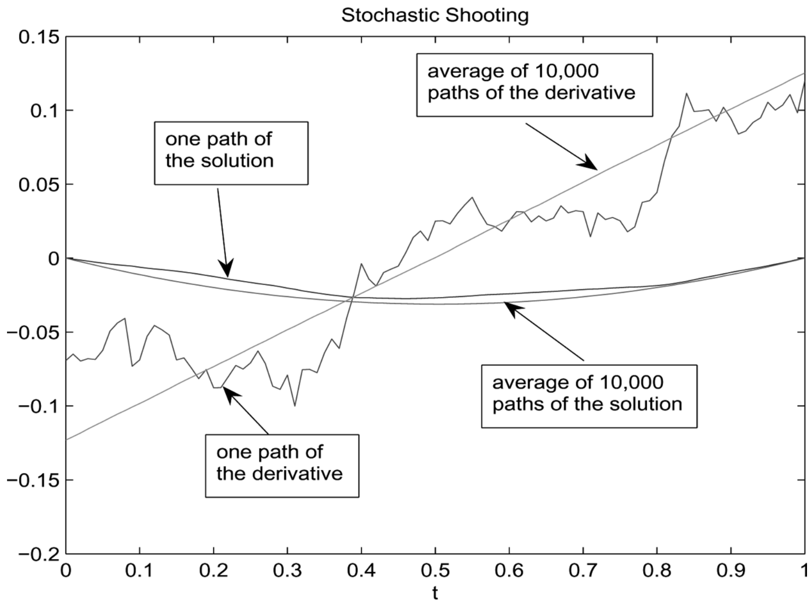

Figure 2. Numerical solution of (5.1) using the shooting method procedure.

This is an example of the temperature distribution in a rod of length unity that has radom heat source. Table 2 presents approximations of $E(X_1(1/2))$ and $E(X_1^2(1/2))$ using the shooting method procedure. The approximate values are based on 10,000 independent trials.

Figure 3 illustrates the average of the approximate solution with 10,000 independent trials using the shooting method procedure. A single trajectory of the numerical solution is also shown.

Compare the results of this example with the results of the similar Example 4.1 in Arciniega and Allen [3]. That is, this shooting

Table 2. Approximate values of $E(X_1(1/2))$ and $E(X_1^2(1/2))$

Number of Intervals in t	$E(X_1(1/2))$	$E(X_1^2(1/2))$
2	0.124782	0.015885
4	0.115755	0.013606
8	0.113888	0.013153
16	0.113243	0.013004
32	0.113333	0.013017
64	0.113365	0.013024

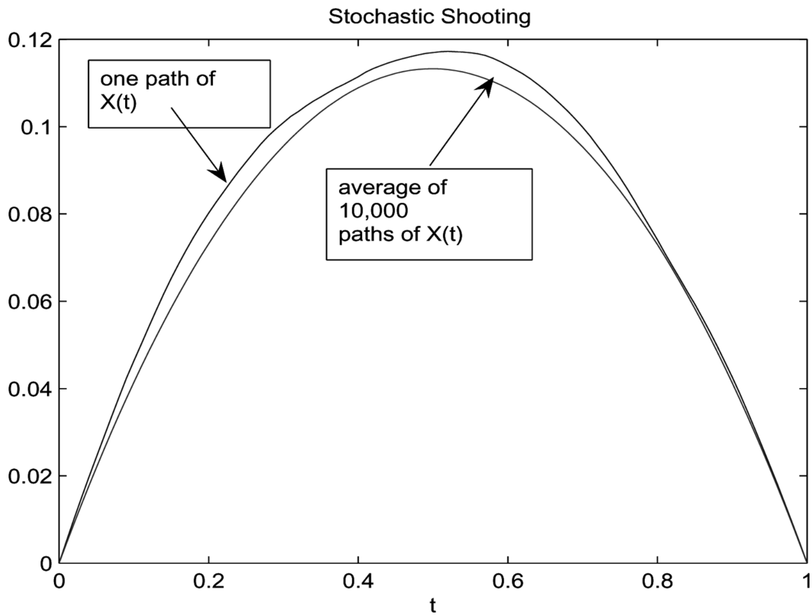

Figure 3. Numerical solution of (5.2) using the shooting method procedure.

method procedure is applicable to both, linear and nonlinear stochastic boundary-value problems.

Finally, consider the following stochastic system:

$$\begin{cases} dX_1(t) = X_2(t)dt + 0.1dW_1(t) + 0.1dW_2(t), \quad 0 \le t \le 1 \\ dX_2(t) = X_1(t)dt + 0.1dW_1(t) + 0.1dW_2(t) \\ X_1(0) = 1 \\ X_2(1) = 0 \end{cases} \tag{5.3}$$

This example has the form (2.1) with

$$\vec{f}(t, \vec{X}(t)) = \begin{bmatrix} -X_2(t) \\ -X_1(t) \end{bmatrix}, \quad M = \begin{bmatrix} .1 & .1 \\ .1 & .1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \vec{\beta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Figure 4 illustrates the average of the approximate solution with 10, 000 independent trials using the shooting method procedure. A single trajectory of the numerical solution also is shown.

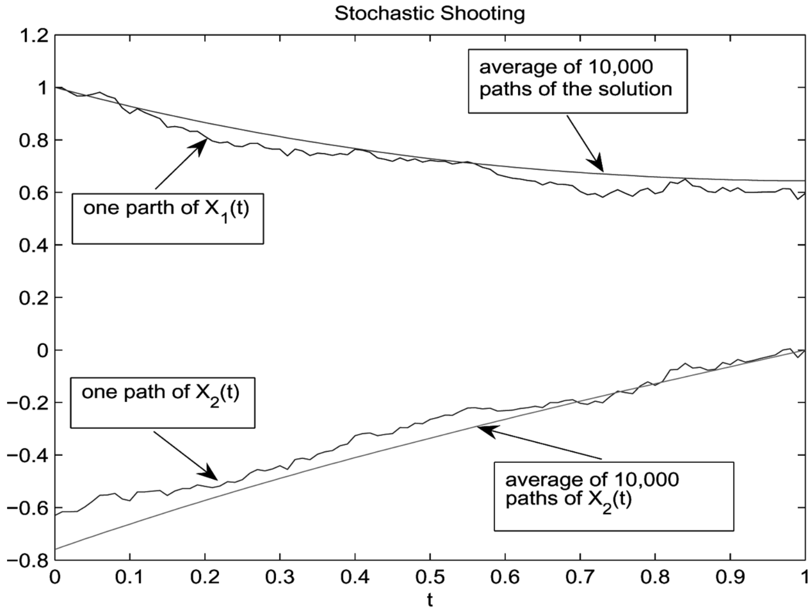

Figure 4. Numerical solution of (5.3) using the shooting method procedure.

6. SUMMARY AND CONCLUSIONS

Stochastic shooting methods were described for numerically solving nonlinear stochastic boundary-value problems. It was shown, trough an error analysis and computational simulations, that these numerical methods provide accurate approximations.

REFERENCES

1. Allen, E.J., and Nunn, C.J. 1995. Difference methods for numerical solution of stochastic two-point boundary-value problems. In: *Proceedings of the First International Conference on Difference Equations*. Elydi, S. N., Greef, J. R., Ladas, G., Peterson, A. C. (eds)., Gordan and Breach Publishers, Amsterdam.
2. Arciniega, A., and Allen, E. 2003. Rounding error in numerical solution of stochastic differential equations. *Stoch. Anal. Appl.* 21(2):281–300.
3. Arciniega, A., and Allen, E. 2004. Shooting methods for numerical solution of stochastic boundary-value problems. *Stoch. Anal. Appl.* 22(5):1295–1314.
4. Arnold, L. 1974. *Stochastic Differential Equations: Theory and Applications*. John Wiley & Sons, New York.
5. Gard, T.C. 1988. *Introduction to Stochastic Differential Equations*. Marcel Dekker, New York.

6. Keller, H.B. 1968. *Numerical Methods for Two-Point Boundary-Value Problems*. Blaisdell Publishing Co., Massachusetts.
7. Kloeden, P.E., and Platen, E. 1992. *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag, New York.
8. Kloeden, P.E., Platen, E., and Schurz, H. 1994. *Numerical Solution of SDE Through Computer Experiments*. Springer-Verlag, Berlin.
9. Milstein, G.N., and Tretyakov, M.V. 2004. *Stochastic Numerics for Mathematical Physics*. Springer-Verlag, Berlin.
10. Nualart, D., and Pardoux, E. 1991. Boundary value problems for stochastic differential equations. *The Annals of Probability* 19(4):1118–1144.
11. Nualart, D., and Pardoux, E. 1991. Second order stochastic differential equations with Dirichlet boundary conditions. *Stochastic Process Appl.* 39(1):1–24.
12. Ocone, D., and Pardoux, E. 1989. Linear stochastic differential equations with boundary conditions. *Probability Theory and Related Fields* 82:489–526.
13. Ortega, J.M. 1972. *Numerical Analysis: A Second Course*. Academic Press, Inc., London.
14. Sharp, W.D., and Allen, E.J. 2000. Stochastic neutron transport equations for rod and plane geometries. *Annals of Nuclear Energy* 27(2):99–116.
15. Talay, D. 1995. Simulation and numerical analysis of stochastic differential systems: a review. In: *Probabilistic Methods in Applied Physics*. Kree, P., Wedig, W., (eds). Lecture Notes in Physics, Springer-Verlag, New York, 451, 63–106.
16. Talay, D., and Turbano, L. 1990. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stoch. Anal. Appl.* 8(4):483–509.
17. Zeitouni, O., and Dembo, A. 1990. A change of variables formula for stratonovich integrals and existence of solutions for two-point stochastic boundary value problems. *Probability Theory and Related Fields* 84:411–425.