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Shooting Methods for Numerical Solution of Stochastic Boundary-Value Problems

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ABSTRACT

In the present investigation, numerical methods are developed for approximate solution of stochastic boundary-value problems. In particular, shooting methods are examined for numerically solving systems of Stratonovich boundary-value problems. It is proved that these methods accurately approximate the solutions of stochastic boundary-value problems. An error analysis of these methods is performed. Computational simulations are given.

Key Words: Shooting methods; Numerical solutions; Stochastic boundary-value problems; Ito and Stratonovich stochastic differential equations.

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1. INTRODUCTION

Methods for numerically solving stochastic initial-value problems have been under much study (see, for example, Refs.[3–5,8,9] and the references therein). However, the theory and numerical solution of stochastic boundary-value problems have received less attention. In the present investigation, shooting methods are applied to numerically solve systems of Stratonovich boundary-value problems. The following linear stochastic system with boundary conditions is the object of interest in the present investigation:

$$\begin{cases} d\vec{u}(t) = (A\vec{u}(t) + \vec{a}(t))dt + \sum_{i=1}^{k}(B_i\vec{u}(t) + \vec{b}_i(t)) \circ dW_i(t), \quad 0 \le t \le 1 \\ F_0\vec{u}(0) + F_1\vec{u}(1) = \vec{0}, \end{cases} \tag{1.1}$$

where $\vec{u}$, $\vec{a}$, $\vec{b}_i \in \mathbb{R}^n$ and A, F_0, F_1, B_i are $n \times n$ matrices, and where the stochastic integrals for this problem are understood in the sense of Stratonovich integrals. In addition, it is assumed that

$$F_0 = \begin{bmatrix} \widehat{F}_0 \\ 0 \end{bmatrix}$$

and

$$F_1 = \begin{bmatrix} 0 \\ \widehat{F}_1 \end{bmatrix}$$

where $\widehat{F}_0$ is a $l \times n$ matrix of rank l and $\widehat{F}_1$ is a $(n-l) \times n$ matrix of rank $n-l$. Also, $0 < l < n$. Ocone and Pardoux[6] and Zeitouni and Dembo[10] have established existence and uniqueness of solutions to Eq. (1.1). As defined in Ref.[10], $\vec{u} \in \mathbb{R}^n$ is a solution to Eq. (1.1) if $\vec{u}$ is Stratonovich integrable and satisfies for all $t \in [0, 1]$:

$$\begin{cases} \vec{u}(t) - \vec{u}(0) = \int_0^t (A\vec{u}(s) + \vec{a}(s))ds + \int_0^t \sum_{i=1}^{k}(B_i\vec{u}(s) + \vec{b}_i(s)) \circ dW_i(s), \\ F_0\vec{u}(0) + F_1\vec{u}(1) = \vec{0}. \end{cases} \tag{1.2}$$

System (1.1) is an anticipative problem as the solution at any position is dependent on the Brownian motion beyond that position. However, Ocone and Pardoux[6] and Zeitouni and Dembo[10] show that solutions to Eq. (1.1) can be defined as standard Stratonovich-type stochastic integrals. In the present investigation, shooting methods are applied to numerically solve Eq. (1.1) and error analyses are performed. A much simpler two-dimensional form of Eq. (1.1) was investigated by Allen and Nunn[1] who studied shooting and finite-difference numerical schemes for approximating the solution.

A shooting method for numerically approximating the solution of Eq. (1.1) is now described. To describe this shooting method, consider the stochastic initial-value system:

$$d\vec{u}_m(t) = (A\vec{u}_m(t) + \vec{a}(t))dt + \sum_{i=1}^{k}(B_i\vec{u}_m(t) + \vec{b}_i(t)) \circ dW_i(t), \tag{1.3}$$

for $m = 1, 2, \dots, n-l+1$, where $\vec{u}_1(0) = \vec{0}$, and $\vec{u}_m(0)$ for $m = 2, 3, \dots, n-l+1$ are chosen to be $n-l$ linearly independent vectors in the null space of F_0. The stochastic system (1.3) is a system of initial-value problems rather than a boundary-value problem. In effect, (1.1) is replaced by system (1.3). The $\vec{u}_m(t)$ obtained using (1.3) can be combined to form a solution to Eq. (1.1). To see this, let

$$\vec{u}(t) = \sum_{m=1}^{n-l+1} \lambda_m \vec{u}_m(t), \tag{1.4}$$

where the λ_m, $1 \le m \le n-l+1$, satisfy

$$\sum_{m=1}^{n-l+1} \lambda_m = 1 \tag{1.5}$$

and the linear system of rank $(n-l)$:

$$\sum_{m=1}^{n-l+1} \lambda_m (F_0\vec{u}_m(0) + F_1\vec{u}_m(1)) = \sum_{m=1}^{n-l+1} \lambda_m F_1 \vec{u}_m(1) = \vec{0}. \tag{1.6}$$

In the next section, it is shown that the solution $\vec{u}(t)$, of (1.1) can be written as given by (1.4) where the $\vec{u}_m(t)$, for $m = 1, 2, \dots, n-l+1$, satisfy (1.3).

As system (1.3) is a stochastic initial-value system, standard numerical methods such as Euler's or Milstein's method (see for example Refs.[3,4]) can be applied to approximate the solution of (1.3) at discrete times. Using (1.5) and (1.6), the values of λ_m can be calculated uniquely and the solution $\vec{u}(t)$ can be approximated by combining the approximate solutions using (1.4). This approach is described in the third section along with an error analysis. Finally, computational results are given to illustrate the procedure.

2. SOLVABILITY USING THE SHOOTING METHOD

In this section, it is verified that the shooting method procedure yields the solution to the original stochastic boundary-value problem (1.1). Clearly, if $\vec{u}(t)$ is given by (1.4), then $\vec{u}(t)$ satisfies (1.1) as

$$\begin{aligned} d\vec{u}(t) &= \sum_{m=1}^{n-l+1} \lambda_m \, d\vec{u}_m(t) \\ &= \sum_{m=1}^{n-l+1} \lambda_m \left((A\vec{u}_m(t) + \vec{a}(t))dt + \sum_{i=1}^{k} (B_i\vec{u}_m(t) + \vec{b}_i(t)) \circ dW_i(t) \right) \\ &= (A\vec{u}(t) + \vec{a}(t))dt + \sum_{i=1}^{k} (B_i\vec{u}(t) + \vec{b}_i(t)) \circ dW_i(t), \end{aligned}$$

where (1.5) has been applied, that is,

$$\sum_{m=1}^{n-l+1} \lambda_m = 1.$$

Also, the boundary conditions are satisfied as:

$$\begin{aligned} F_0\vec{u}(0) + F_1\vec{u}(1) &= \sum_{m=1}^{n-l+1} \lambda_m (F_0\vec{u}_m(0) + F_1\vec{u}_m(1)) \\ &= \sum_{m=1}^{n-l+1} \lambda_m F_1 \vec{u}_m(1) = \vec{0}, \end{aligned}$$

as $\vec{u}_1(0) = \vec{0}$ and $\vec{u}_m(0)$ are linearly independent vectors in the null space of F_0 for $m = 2, 3, \ldots, n-l+1$, and $\sum_{m=1}^{n-l+1} \lambda_m F_1 \vec{u}_m(1) = \vec{0}$. The next theorem shows that the λ_m, $1 \leq m \leq n-l+1$, satisfying Eqs. (1.5) and (1.6) can be uniquely determined. Thus, the solution of the original stochastic boundary-value problem (1.1) is obtained using (1.4) by combining the solutions $\vec{u}_m(t)$ of the initial-value problems (1.3).

Theorem 2.1. *The* λ_m, $1 \le m \le n - l + 1$ *uniquely exist and satisfy Eqs.* (1.5) *and* (1.6).

Proof. For the original stochastic boundary-value problem:

$$\begin{cases} d\vec{u}(t) = (A\vec{u}(t) + \vec{a}(t))dt \\ \qquad\qquad + \sum_{i=1}^{k}(B_i\vec{u}(t) + \vec{b}_i(t)) \circ dW_i(t), \quad 0 \le t \le 1 \\ F_0\vec{u}(0) + F_1\vec{u}(1) = \vec{0}, \end{cases} \tag{2.1}$$

the solution $\vec{u}(t)$ uniquely exists, which is proved in Refs.[6,10]. Consider the corresponding stochastic initial-value problem:

$$\begin{cases} d\vec{u}_1(t) = (A\vec{u}_1(t) + \vec{a}(t))dt \\ \qquad\qquad + \sum_{i=1}^{k}(B_i\vec{u}_1(t) + \vec{b}_i(t)) \circ dW_i(t), \quad 0 \le t \le 1 \\ \vec{u}_1(0) = \vec{0}, \end{cases} \tag{2.2}$$

It is well-known that $\vec{u}_1(t)$ is uniquely determined for this problem (see, for example, Refs.[2–4]). Subtracting Eqs. (2.1) and (2.2) yields:

$$\begin{cases} d\vec{w}(t) = A\vec{w}(t)dt + \sum_{i=1}^{k}(B_i\vec{w}(t)) \circ dW_i(t), \quad 0 \le t \le 1 \\ F_0\vec{w}(0) + F_1\vec{w}(1) = -F_1\vec{u}_1(1), \end{cases} \tag{2.3}$$

where $\vec{w}(t) = \vec{u}(t) - \vec{u}_1(t)$. As $\vec{u}(t)$ and $\vec{u}_1(t)$ uniquely exist, then $\vec{w}(t)$ also uniquely exists. In particular, $\vec{w}(0) = \vec{u}(0)$ exists. Consider, next

$$F_0\vec{w}(0) = \begin{bmatrix} \widehat{F}_0 \\ 0 \end{bmatrix} \quad \vec{w}(0) = \begin{bmatrix} \widehat{F}_0\vec{w}(0) \\ \vec{0} \end{bmatrix}$$

and

$$F_1\vec{w}(1) = \begin{bmatrix} 0 \\ \widehat{F}_1 \end{bmatrix} \quad \vec{w}(1) = \begin{bmatrix} \vec{0} \\ \widehat{F}_1\vec{w}(1) \end{bmatrix}.$$

Then,

$$F_0\vec{w}(0) + F_1\vec{w}(1) = \begin{bmatrix} \widehat{F}_0\vec{w}(0) \\ \widehat{F}_1\vec{w}(1) \end{bmatrix} = \begin{bmatrix} \vec{0} \\ -\widehat{F}_1\vec{u}_1(1) \end{bmatrix}.$$

Thus $F_0\vec{w}(0) = \vec{0}$ and $F_1\vec{w}(1) = -F_1\vec{u}_1(1)$. Hence $\vec{w}(0)$ is in the null space of F_0. However, the null space of F_0 and the range of F_1 are identical, i.e., $\mathrm{Null}(F_0) = \mathrm{Range}(F_1)$. Furthermore, $\dim(\mathrm{Range}(F_1)) = n - l = \dim(\mathrm{Null}(F_0))$. Therefore, one can let

$$\vec{w}(0) = \sum_{m=2}^{n-l+1} \lambda_m \vec{v}_m(0),$$

where $\vec{v}_m(0) = \vec{u}_m(0)$ are linearly independent vectors in the range of F_1 and $\{\lambda_m\}_{m=2}^{n-l+1}$ are to be determined. However, as $\vec{w}(0) \in \mathrm{Null}(F_0)$, $\{\lambda_m\}_{m=2}^{n-l+1}$ uniquely exist. Consider now the stochastic initial-value problem:

$$\begin{cases} d\vec{w}(t) = A\vec{w}(t)dt + \displaystyle\sum_{i=1}^{k}(B_i\vec{w}(t)) \circ dW_i(t), & 0 \le t \le 1 \\ \vec{w}(0) = \displaystyle\sum_{m=2}^{n-l+1} \lambda_m \vec{v}_m(0). \end{cases} \tag{2.4}$$

Therefore, (2.4) has a unique solution. As (2.3) and (2.4) have unique solutions, the solutions agree by the above argument. Now consider $\vec{v}_m(t) = \vec{u}_m(t) - \vec{u}_1(t)$ that solves the stochastic initial-value problem:

$$\begin{cases} d\vec{v}_m(t) = A\vec{v}_m(t)dt + \displaystyle\sum_{i=1}^{k}(B_i\vec{v}_m(t)) \circ dW_i(t), & 0 \le t \le 1 \\ F_0\vec{v}_m(0) = \vec{0}, \end{cases} \tag{2.5}$$

for $m = 2, 3, \ldots, n - l + 1$. Then, it is clear that

$$\vec{w}(t) = \sum_{m=2}^{n-l+1} \lambda_m \vec{v}_m(t).$$

Thus, as $F_1\vec{w}(1) = -F_1\vec{u}_1(1)$, then $\lambda_2, \lambda_3, \ldots, \lambda_{n-l+1}$ satisfy $F_1 \times \sum_{m=2}^{n-l+1} \lambda_m \vec{v}_m(1) = -F_1\vec{u}_1(1)$. Finally, notice that

$$\vec{u}(t) = \vec{w}(t) + \vec{u}_1(t).$$

Therefore,

$$\vec{u}(t) = \vec{u}_1(t) + \sum_{m=2}^{n-l+1} \lambda_m \vec{v}_m(t) = \vec{u}_1(t) + \sum_{m=2}^{n-l+1} \lambda_m(\vec{u}_m(t) - \vec{u}_1(t)).$$

Thus,

$$\vec{u}(t) = \sum_{m=1}^{n-l+1} \lambda_m \vec{u}_m(t),$$

where

$$F_1 \sum_{m=2}^{n-l+1} \lambda_m(\vec{u}_m(1) - \vec{u}_1(1)) = -F_1\vec{u}_1(1),$$

$$\sum_{m=1}^{n-l+1} \lambda_m = 1,$$

and $F_0\vec{v}_m(0) = \vec{0}$ for each $m = 1, 2, \ldots, n - 1 + l$. Thus, λ_m, $m = 1, 2, \ldots, n - l + 1$, uniquely exist and satisfy (1.5) and (1.6). □

3. NUMERICAL SOLUTION AND ERROR ANALYSIS

In this section, error analyses for numerical solution of system (1.3) are performed. Then, it is verified that Eq. (1.4) yields correspondingly accurate approximations to $\vec{u}(t)$ in the original stochastic boundary-value problem (1.1). Two numerical methods are considered to numerically solve system (1.3), namely, Euler's method and Milstein's method. The approximate solutions obtained by numerically solving (1.3) are then combined to approximate the solution of (1.1) using (1.4).

To perform an error analysis, it is useful to convert the Stratonovich system (1.3) to its corresponding Ito system. The Ito form of system (1.3) is given by (see Ref.[4]):

$$d\vec{u}_m(t) = \left\{(A\vec{u}_m(t) + \vec{a}(t)) + \frac{1}{2}\sum_{j=1}^{k}(B_j^2\vec{u}_m(t) + B_j\vec{b}_j(t))\right\}dt + \sum_{i=1}^{k}(B_i\vec{u}_m(t) + \vec{b}_i(t))dW_i(t). \tag{3.1}$$

System (3.1) is solved numerically rather than (1.3). By solving (3.1) one solves (1.3) as the two systems are equivalent. To solve (3.1) numerically, select a positive integer $N \geq 2$ and partition the interval $[0, 1]$ into

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

where $t_p = ph$ for each $p = 0, 1, \ldots, N$. It is assumed that the step size h is fixed, so that the common distance between the discrete times is $h = \frac{1}{N}$. For example, the Euler approximations to system (3.1) are stochastic processes satisfying the iterative scheme (see Refs.[3,4]):

$$\vec{u}_{m,p+1} = \vec{u}_{m,p} + h\left\{ (A\vec{u}_{m,p} + \vec{a}(t_p)) + \frac{1}{2}\sum_{j=1}^{k}(B_j^2\vec{u}_{m,p} + B_j\vec{b}_j(t_p)) \right\} + \sum_{i=1}^{k}(B_i\vec{u}_{m,p} + \vec{b}_i(t_p))\sqrt{h}\eta_i, \tag{3.2}$$

for each $m = 1, 2, \ldots, n - l + 1$. In the above scheme, $\vec{u}_{m,p}$ denotes the approximation to the exact solution at the pth time step. That is $\vec{u}_{m,p} \approx \vec{u}_m(t_p)$, for each $p = 0, 1, \ldots, N - 1$, $m = 1, 2, \ldots, n - l + 1$. Also, the random increments η_i are independent normal random variables with mean zero and variance unity, i.e., $\eta_i \in N(0, 1)$ (see Ref.[4]). After (3.2) is solved numerically for each $m = 1, 2, \ldots, n - l + 1$, the approximate $\hat{\lambda}_m$, for $m = 1, 2, \ldots, n - l + 1$, are calculated using

$$\sum_{m=1}^{n-l+1} \hat{\lambda}_m (F_0\vec{u}_{m,N} + F_1\vec{u}_{m,N}) = \sum_{m=1}^{n-l+1} \hat{\lambda}_m F_1\vec{u}_{m,N} = \vec{0}$$

with

$$\sum_{m=1}^{n-l+1} \hat{\lambda}_m = 1$$

corresponding to (1.5) and (1.6). As a result,

$$\vec{u}(t_p) \approx \sum_{m=1}^{n-l+1} \hat{\lambda}_m \vec{u}_{m,p}$$

corresponding to (1.4).

The theorem below is a well-known result concerning the strong convergence of Euler's method for stochastic differential equations (see Refs.[3,4,8]). To be consistent with existing literature, the following notation is used in the present investigation. In particular, denote

$$\vec{f}(t, \vec{u}_m(t)) = (A\vec{u}_m(t) + \vec{a}(t)) + \frac{1}{2}\sum_{j=1}^{k}(B_j^2\vec{u}_m(t) + B_j\vec{b}_j(t))$$

and

$$G(t, \vec{u}_m(t)) = \sum_{i=1}^{k}(B_i\vec{u}_m(t) + \vec{b}_i(t)).$$

Then, the system

$$d\vec{u}_m(t) = \vec{f}(t, \vec{u}_m(t))dt + G(t, \vec{u}_m(t))d\vec{W}(t)$$

is equivalent to (3.1). In the above system,

$\vec{f} = \{f_i\}$ is an n-vector-valued function,
$G = \{g_{i,j}\}$ is an $n \times k$-matrix-valued function,
$\vec{W} = \{W_i\}$ is a k-dimensional Wiener process,

and the solution $\vec{u}_m$ is an n-dimensional process. The above system can be expressed as

$$d\vec{u}_m(t) = \vec{f}(t, \vec{u}_m(t))\, dt + \sum_{i=1}^{k} \vec{g}_i(t, \vec{u}_m(t))\, dW_i(t),$$

where the $\vec{g}_i$ are the columns of the matrix G and the W_i are the independent scalar Wiener processes forming the components of $\vec{W}$.

Theorem 3.1. *Consider the system of Ito stochastic differential equations,*

$$d\vec{u}_m(t) = \vec{f}(t, \vec{u}_m(t))dt + G(t, \vec{u}_m(t))d\vec{W}(t)$$

for $m = 1, 2, \ldots, n - l + 1$, *where* $\vec{f} \in \mathbb{R}^n$, $G \in \mathbb{R}^{n \times k}$, *and* $\vec{W} \in \mathbb{R}^k$. *Suppose* $\vec{f}$ *and* G *satisfy uniform growth and Lipschitz conditions in the second variable, and are Hölder continuous of order* $\frac{1}{2}$ *in the first variable. Specifically, there exists a constant* $K_m > 0$ *for each* $m = 1, \ldots, n - l + 1$ *such that for all* $s, t \in [0, 1]$, $\vec{u}_m, \vec{v}_m \in \mathbb{R}^n$,

$$\begin{aligned} ||\vec{f}(t, \vec{u}_m(t)) - \vec{f}(t, \vec{v}_m(t))|| + ||G(t, \vec{u}_m(t)) - G(t, \vec{v}_m(t))|| \\ \leq K||\vec{u}_m - \vec{v}_m|| \end{aligned} \tag{3.3}$$

$$||\vec{f}(t, \vec{u}_m(t))||^2 + ||G(t, \vec{u}_m(t))||^2 \leq K^2(1 + ||\vec{u}_m||^2) \tag{3.4}$$

$$||\vec{f}(s, \vec{u}_m(t)) - \vec{f}(t, \vec{u}_m(t))|| + ||G(s, \vec{u}_m(t)) - G(t, \vec{u}_m(t))|| \leq K||s - t||^{\frac{1}{2}}. \tag{3.5}$$

Then, there exists a positive constant C_m *such that*

$$E||\vec{u}_m(t_p) - \vec{u}_{m,p}||^2 \leq C_m h$$

for each $m = 1, 2, \ldots, n - l + 1$, *where* $||\cdot||$ *is the Euclidean norm.*

In Theorem 3.1, the Eqs. (3.3) and (3.4) guarantee existence and uniqueness of solutions of the Ito stochastic differential equations and equation (3.5) guarantees the convergence of the Euler method (see Refs.[2–4]).

A second method to numerically approximate the solution of system (3.1) is the Milstein method (see Ref.[4]):

$$\begin{aligned}\vec{u}_{m,p+1} = \vec{u}_{m,p} + h\bigg\{(A\vec{u}_{m,p} + \vec{a}(t_p)) + \frac{1}{2}\sum_{j=1}^{k}(B_j^2\vec{u}_{m,p} + B_j\vec{b}_j(t_p))\bigg\}\\ + \sum_{i=1}^{k}(B_i\vec{u}_{m,p} + \vec{b}_i(t_p))\sqrt{h}\eta_i + \sum_{j_1,j_2}^{k} I_{(j_1,j_2)}\big[B_{j_2}B_{j_1}\vec{u}_{m,p} + B_{j_2}\vec{b}_{j_1}\big],\end{aligned} \tag{3.6}$$

where

$$\begin{cases} I_{(j_1,j_2)} = \displaystyle\int_{\tau_n}^{\tau_{n+1}}\int_{\tau_n}^{s_1} dW_{s_2}^{j_1}\, dW_{s_1}^{j_2}, \quad j_1 \neq j_2 \\ I_{(j_1,j_1)} = \dfrac{1}{2}\big\{(\Delta W_{j_1})^2 - h\big\}. \end{cases} \tag{3.7}$$

The last term in Eq. (3.6) differentiates Milstein's method from Euler's method. Notice that Milstein's method is complicated for general problems, due to the evaluation of $I_{(j_1,j_2)}$. If $\vec{b}_j(t) = \vec{0}$ for $j = 1, 2, \ldots, k$ and $B_{j_2}B_{j_1} = B_{j_1}B_{j_2}$ for $j_1, j_2 = 1, 2, \ldots, k$, then Milstein's method becomes:

$$\begin{aligned}\vec{u}_{m,p+1} = \vec{u}_{m,p} + h\big\{(A\vec{u}_{m,p} + \vec{a}(t_p))\big\} + \sum_{i=1}^{k} B_i\vec{u}_{m,p}\sqrt{h}\eta_i\\ + \frac{1}{2}\sum_{j_1}^{k}\sum_{j_2}^{k}(\Delta W_{j_1})(\Delta W_{j_2})B_{j_2}B_{j_1}\vec{u}_{m,p},\end{aligned} \tag{3.8}$$

and the evaluation of $I_{(j_1,j_2)}$ is not required. The following theorem states that Milstein's method has second-order strong convergence in the mean square error as compared with Euler's method, which has first-order strong convergence (see Refs.[3,4]).

Theorem 3.2. *Under the hypotheses of Theorem* 3.1 *for the Milstein approximations* (3.6), *the following estimate holds*:

$$E||\vec{u}_m(t_p) - \vec{u}_{m,p}||^2 \leq C_m h^2$$

for each $m = 1, 2, \ldots, n - l + 1$.

In the next two theorems, it is shown that for the shooting method the errors between the exact and approximate solutions are small.

Theorem 3.3. *The error in estimating* $\vec{\lambda}$ *is of the same order as the error in estimating the solutions of the initial-value problems.*

Proof. Consider the equations

$$\sum_{m=1}^{n-l+1} \lambda_m F_1 \vec{u}_m(1) = \vec{0}, \tag{3.9}$$

and,

$$\lambda_1 = 1 - \sum_{m=2}^{n-l+1} \lambda_m. \tag{3.10}$$

Substituting Eq. (3.10) into Eq. (3.9) and rearranging terms yields

$$\sum_{m=2}^{n-l+1} \lambda_m F_1 (\vec{u}_m(1) - \vec{u}_1(1)) = -F_1 \vec{u}_1(1). \tag{3.11}$$

But

$$F_1 = \begin{bmatrix} 0 \\ \widehat{F}_1 \end{bmatrix}$$

where $\widehat{F}_1$ is a $(n - l) \times n$ matrix of rank $n - l$. Thus, the equation in (3.11) is a linear system of the form

$$A\vec{\lambda} = \vec{b}, \tag{3.12}$$

where A is an $(n - l) \times (n - l)$ matrix, $\vec{\lambda}$ and $\vec{b}$ are vectors of length $n - l$. In particular, $\vec{b} = -\widehat{F}_1 \vec{u}_1(1)$ and $A = [\vec{a}_1, \ldots, \vec{a}_{n-l}]$, where

$\vec{a}_i = \widehat{F}_1[\vec{u}_{i+1}(1) - \vec{u}_1(1)]$. Then,

$$\vec{\lambda} = A^{-1}\vec{b}. \tag{3.13}$$

In numerical solution of the initial-value problem (3.1) the $\vec{u}_m(1)$, for $m = 2, \ldots, n-l+1$ are being approximated by $\hat{\vec{u}}_m(1)$. Therefore, the linear system to be solved is an approximation of (3.12). Call this linear system

$$\widehat{A}\hat{\vec{\lambda}} = \hat{\vec{b}}, \tag{3.14}$$

where $\widehat{A}=[\hat{\vec{a}}_1,\ldots,\hat{\vec{a}}_{n-l}]$, $\hat{\vec{a}}_i=\widehat{F}_1[\hat{\vec{u}}_{i+1}-\hat{\vec{u}}_1(1)]$, and $\hat{\vec{b}}=-\widehat{F}_1\hat{\vec{u}}_1(1)$. However, notice by Theorems 3.1 and 3.2, that $\widehat{A}$ and $\hat{\vec{b}}$ are perturbations to A and $\vec{b}$ for small time step h. It is well-known (see, e.g., Ref.[7]) that

$$||\vec{\lambda} - \hat{\vec{\lambda}}|| \leq ||\vec{\lambda}|| \left[\frac{||A||||A^{-1}||}{1 - ||A^{-1}||||A - \widehat{A}||}\right] \left[\frac{||A - \widehat{A}||}{||A||} + \frac{||\vec{b} - \hat{\vec{b}}||}{||\vec{b}||}\right].$$

Hence,

$$\begin{aligned} E||\vec{\lambda} - \hat{\vec{\lambda}}|| \leq {} & \left(E\left[\frac{||\vec{\lambda}||||A^{-1}||}{1 - ||A^{-1}||||A - \widehat{A}||}\right]^2\right)^{1/2} \left(E||A - \widehat{A}||^2\right)^{1/2} \\ & + \left(E\left[\frac{||\vec{\lambda}||||A||||A^{-1}||}{||\vec{b}||(1 - ||A^{-1}||||A - \widehat{A}||)}\right]^2\right)^{1/2} \left(E||\vec{b} - \hat{\vec{b}}||^2\right)^{1/2}. \end{aligned}$$

Thus, the error $E||\vec{\lambda} - \hat{\vec{\lambda}}||$ is proportional to the error obtained in estimating $\vec{u}_m(1)$ using, for example, Euler's method or Milstein's method. This completes the proof of the theorem. □

Theorem 3.4. *Let* $\vec{u}(t)$ *be the exact solution of the boundary-value problem* (1.1) *and* $\hat{\vec{u}}(t)$ *the approximate solution obtained using the initial-value system* (3.1). *Then,*

$$\begin{aligned} & E||\vec{u}(t_p) - \hat{\vec{u}}(t_p)||^{1/2} \\ & \quad \leq \left(\sum_{m=1}^{n-l+1} E|\lambda_m|\right)^{1/2} \left(\sum_{m=1}^{n-l+1} E||\vec{u}_m(t_p) - \vec{u}_{m,p}||\right)^{1/2} \\ & \qquad + \left(E||\vec{\lambda} - \hat{\vec{\lambda}}||\right)^{1/2} \left(\sum_{m=1}^{n-l+1} E||\vec{u}_{m,p}||\right)^{1/2}. \end{aligned}$$

Proof. Consider

$$\hat{\vec{u}}(t_p) = \sum_{m=1}^{n-l+1} \hat{\lambda}_m \vec{u}_{m,p} \approx \vec{u}(t_p).$$

Then,

$$\begin{aligned}
\vec{u}(t_p) - \hat{\vec{u}}(t_p) &= \sum_{m=1}^{n-l+1} \left[\lambda_m \vec{u}_m(t_p) - \hat{\lambda}_m \vec{u}_{m,p}\right] \\
&= \sum_{m=1}^{n-l+1} \lambda_m (\vec{u}_m(t_p) - \vec{u}_{m,p}) + \sum_{m=1}^{n-l+1} (\lambda_m - \hat{\lambda}_m)\vec{u}_{m,p}. \qquad (3.15)
\end{aligned}$$

As a result,

$$\begin{aligned}
&E||\vec{u}(t_p) - \hat{\vec{u}}(t_p)||^{1/2} \\
&\leq E\left(\sum_{m=1}^{n-l+1} |\lambda_m|||\vec{u}_m(t_p) - \vec{u}_{m,p}||\right)^{1/2} + E\left(\sum_{m=1}^{n-l+1} |\lambda_m - \hat{\lambda}_m|||\vec{u}_{m,p}||\right)^{1/2} \\
&\leq E\left[\left(\sum_{m=1}^{n-l+1} |\lambda_m|^2\right)^{1/4} \left(\sum_{m=1}^{n-l+1} ||\vec{u}_m(t_p) - \vec{u}_{m,p}||^2\right)^{1/4}\right] \\
&\quad + E\left[\left(\sum_{m=1}^{n-l+1} |\lambda_m - \hat{\lambda}_m|^2\right)^{1/4} \left(\sum_{m=1}^{n-l+1} ||\vec{u}_{m,p}||^2\right)^{1/4}\right] \\
&\leq \left(\left(E\left(\sum_{m=1}^{n-l+1} |\lambda_m|^2\right)^{1/2}\right)\right)^{1/2} \left(E\left(\left(\sum_{m=1}^{n-l+1} ||\vec{u}_m(t_p) - \vec{u}_{m,p}||^2\right)^{1/2}\right)\right)^{1/2} \\
&\quad + \left(E||\vec{\lambda} - \hat{\vec{\lambda}}||\right)^{1/2} \left(E\left(\left(\sum_{m=1}^{n-l+1} ||\vec{u}_{m,p}||^2\right)^{1/2}\right)\right)^{1/2} \\
&\leq \left(\sum_{m=1}^{n-l+1} E|\lambda_m|\right)^{1/2} \left(\sum_{m=1}^{n-l+1} E||\vec{u}_m(t_p) - \vec{u}_{m,p}||\right)^{1/2} \\
&\quad + \left(E||\vec{\lambda} - \hat{\vec{\lambda}}||\right)^{1/2} \left(\sum_{m=1}^{n-l+1} E||\vec{u}_{m,p}||\right)^{1/2}, \qquad (3.16)
\end{aligned}$$

using the Cauchy-Schwarz inequality and the inequality:

$$\sqrt{\sum_{i=1}^{N} |a_i|^2} \leq \sum_{i=1}^{N} |a_i|.$$

This completes the proof. □

Note by Theorems 3.1–3.4, the numerical method is convergent with accuracy determined by the order of accuracy of the methods used to approximately solve the initial-value problems.

4. COMPUTATIONAL RESULTS

In this section, computational results are given to test the numerical method developed in the present investigation. A description of an interesting first problem is presented here (see Ref.[1].) Consider the second-order two-point stochastic boundary-value problem:

$$\begin{cases} \varphi''(x) = (-1 + \varphi(x))dx + \varphi(x) \circ dW(x) \\ \varphi(0) = 0 \\ \varphi(1) = 0. \end{cases} \tag{4.1}$$

Letting $y_1(x) = \varphi(x)$ and $y_2(x) = \varphi'(x)$, Eq. (4.1) becomes:

$$\begin{cases} dy_1(x) = y_2(x)dx \\ dy_2(x) = (-1 + y_1(x))dx + y_1(x) \circ dW(x) \\ y_1(0) = 0 \\ y_1(1) = 0. \end{cases} \tag{4.2}$$

Now, letting $u_1(t) = y_1(x)$, $u_2(t) = y_2(x)$, with $t = x$, Eq. (4.2) becomes:

$$\begin{cases} d\vec{u}(t) = (A\vec{u}(t) + \vec{a}(t))dt \\ \qquad\qquad + \displaystyle\sum_{i=1}^{2}(B_i\vec{u}(t) + \vec{b}_i(t)) \circ dW_i(t), \quad 0 \leq t \leq 1 \\ F_0\vec{u}(0) + F_1\vec{u}(1) = \vec{0}, \end{cases} \tag{4.3}$$

where

$$\vec{a}(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \vec{b}_1 = \vec{0}, \quad \vec{b}_2 = \vec{0}, \quad \text{and} \quad B_2 = 0.$$

Thus, problem (4.3) has the form

$$\begin{cases} d\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right) dt + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \circ dW_1(t) \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases} \tag{4.4}$$

To solve this problem, consider two different solutions to a corresponding stochastic initial-value problem. Consider $\vec{u}_m(t)$, $m = 1, 2$ where

$$\vec{u}_1(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{u}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The $\vec{u}_m(t)$ for $m = 1, 2$ solve the stochastic initial-value problem:

$$\begin{cases} d\vec{u}_m(t) = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}_m(t) \right) dt + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \vec{u}_m(t) \circ dW(t) \\ \quad \text{for } m = 1, 2 \text{ where } \vec{u}_m(0) \text{ is given above .} \end{cases} \tag{4.5}$$

Notice that Eq. (4.5) is solved for both $\vec{u}_1(t)$ and $\vec{u}_2(t)$ using different initial conditions but the same Wiener process. Euler and Milstein methods are used for comparison in numerically solving the corresponding stochastic initial-value problems. After $\vec{u}_1(t)$ and $\vec{u}_2(t)$ are numerically solved to time $t = 1$, they are combined to approximate the solution of the stochastic boundary-value problem (4.4). The numerical results are shown below. For this problem, the Euler and Milstein methods are identical. Also, the Ito form and the Stratonovich form of this problem are the same. Table 4.1 presents approximations of $E(u_1(1/2))$ and $E(u_1^2(1/2))$ using both, the Euler and Milstein methods. The approximate values are based on 100,000 independent trials. Figure 4.1 illustrates the average of the approximate solution with 100,000 independent trials, using both, the Euler and Milstein

Table 4.1. Approximate values of $E(u_1(1/2))$ and $E(u_1^2(1/2))$.

	Euler shooting		Milstein shooting	
Number of intervals in t (h)	Eu	Eu^2	Eu	Eu^2
2	0.1251	0.0156	0.1251	0.0156
4	0.1169	0.0138	0.1169	0.0138
8	0.1155	0.0135	0.1155	0.0135
16	0.1153	0.0135	0.1153	0.0135

Methods. Two particular trajectories of the solutions are also shown. Absolute errors of the numerical solution at time $t = 0.5$ are shown in Figure 4.2 for Euler and Milstein methods.

As a second example, consider the following two-point stochastic boundary-value problem:

$$\begin{cases} d\vec{u}(t) = (A\vec{u}(t) + \vec{a}(t))dt + \sum_{i=1}^{2}(B_i\vec{u}(t) + \vec{b}_i(t)) \circ dW_i(t), & 0 \le t \le 1 \\ F_0\vec{u}(0) + F_1\vec{u}(1) = \vec{0} \end{cases} \tag{4.6}$$

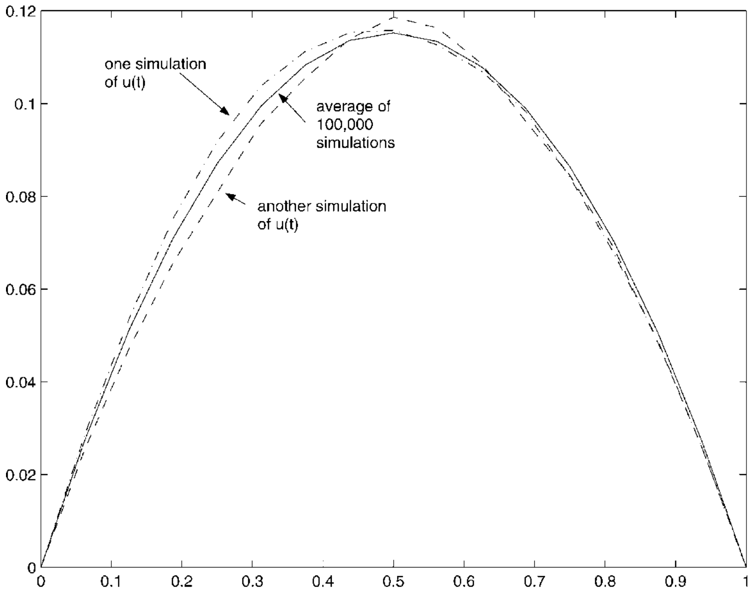

Figure 4.1. Illustration of the average and two trajectories of the solution.

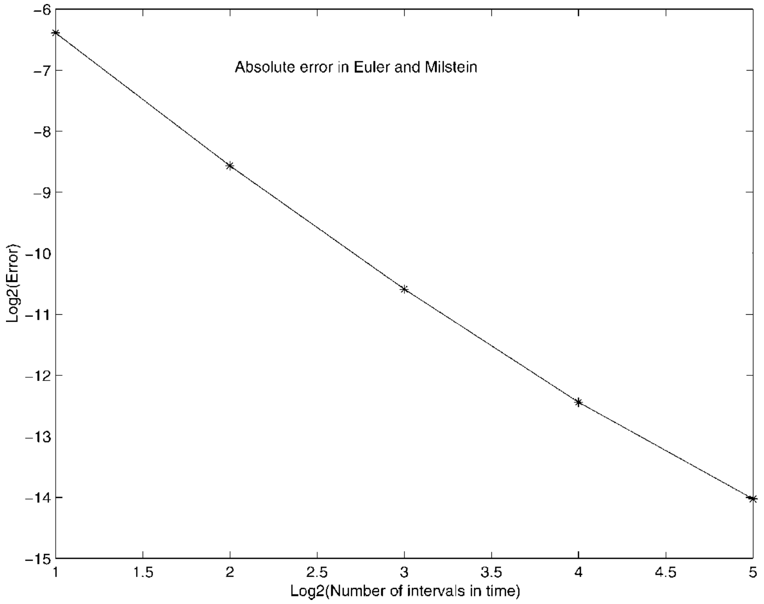

Figure 4.2. Illustration of the absolute errors for the first example.

where

$$\vec{a}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \quad B_1 = \frac{1}{10}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_2 = \frac{1}{10}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\vec{b}_1 = \vec{0}, \quad \vec{b}_2 = \vec{0}.$$

As in the first example, the corresponding initial-value problem is solved numerically using three different initial conditions:

$$\vec{u}_1(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{u}_3(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The three numerical solutions are then combined to approximate the solution to the original boundary-value problem (4.6). Table 4.2 presents

Table 4.2. Approximate values of $E(u_1(1/2))$ and $E(u_1^2(1/2))$.

	Euler shooting		Milstein shooting	
Number of intervals in t (h)	Eu	Eu^2	Eu	Eu^2
2	0.4493	0.2019	0.4491	0.2017
4	0.5256	0.2771	0.5262	0.2777
8	0.5729	0.3305	0.5741	0.3317
16	0.5988	0.3620	0.6002	0.3637
32	0.6118	0.3786	0.6134	0.3806
64	0.6183	0.3872	0.6200	0.3893

approximations of $E(u_1(1/2))$ and $E(u_1^2(1/2))$ using both the Euler and Milstein methods. The approximate values are based on 100,000 independent trials. Notice that only the approximations of the first component of the vector are given. Although the exact values are unknown, the best estimates for $E(u_1(1/2))$ and $E(u_1^2(1/2))$ are, respectively, 0.6248 and 0.3960. Notice that Milstein's method appears to converge

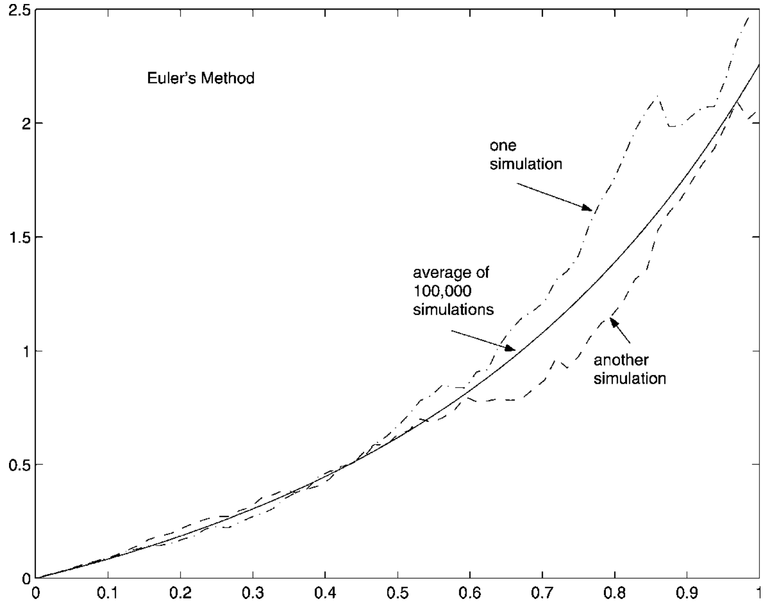

Figure 4.3. Illustration of the average and two trajectories of the solution.

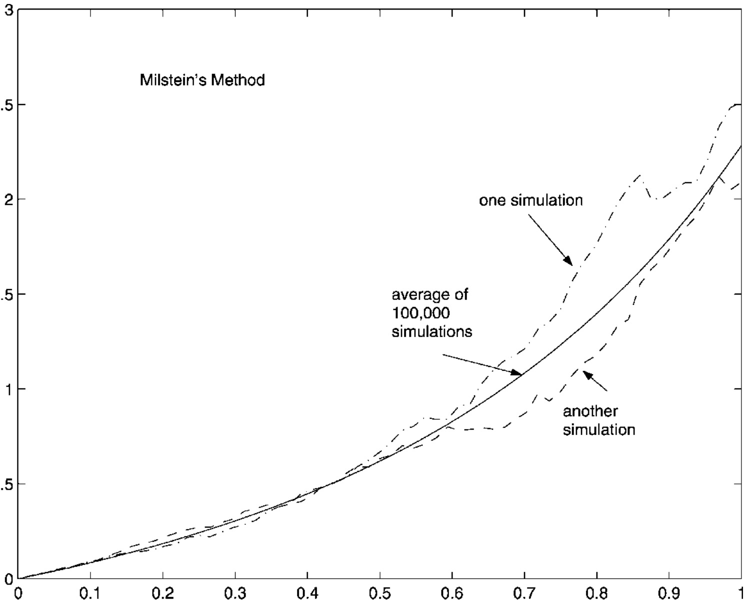

Figure 4.4. Illustration of the average and two trajectories of the solution.

slightly faster than Euler's method for this example. Figure 4.3 illustrates the average of the approximate solution with 100,000 independent trials, using Euler's Method. Two particular trajectories of the solutions are also shown.

Figure 4.4 illustrates the average of the approximate solution with 100,000 independent trials, using Milstein's Method. Two particular trajectories of the solutions are also shown.

5. CONCLUSION

Numerical methods were used to numerically solve a stochastic boundary-value system. In particular, shooting methods were examined for numerically solving systems of Stratonovich boundary-value problems. It was proved that these methods accurately approximate the solutions of stochastic boundary-value problems. Error analyses of these methods were performed. Computational simulations were given.

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