INDISCERNIBLE SEQUENCES IN BANACH SPACE GEOMETRY

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Contents

0. Introduction

The impact of logic in Banach space theory. If one were to assemble a list of the most important results in of the last thirty years in Banach space theory, the following would have to be included.

(1) Tsirelson's example of a Banach space not containing ℓ_p or c_0 [75];
(2) Krivine's Theorem [49];
(3) The Krivine-Maurey theorem that every stable space contains some ℓ_p almost isometrically [50];
(4) The Bourgain-Rosenthal-Schechtman theorem that there are uncountably many complemented subspaces of L_p [3];
(5) Gowers' dichotomy [22].

Apart from their importance, these results have in common that they were proved by borrowing concepts and techniques from logic. Tsirelson's construction was inspired by the method of forcing; Krivine's theorem was proved using ultraproducts and compactness; the Krivine-Maurey theorem was based on the notion of model theoretic stability; the main tool of the Bourgain-Rosenthal-Schechtman paper is an ordinal rank of the type commonly used in model theory; and Gowers dichotomy was proved using Gowers' celebrated block Ramsey theorem [24], which is an elaboration of the Galvin-Prikry proof [17] of Silver's theorem that every analytic set is Ramsey [72].

By all accounts, one of the most elegant theorems of modern Banach space theory is Rosenthal's ℓ_1 theorem [64]. It was observed by Farahat [14] that Rosenthal's proof contains an independent proof of the classical theorem that every closed subset of $\mathbb{N}^{\mathbb{N}}$ is Ramsey. This observation unveiled infinite Ramsey theory as a chief tool in Banach space theory (for a rather old but excellent survey, see [54]) and triggered a host of applications that culminated with Gower's famous dichotomy [22].

The case of model theory. A close analysis of the concepts and techniques that have played an important role in the development of modern Banach space theory will reveal a striking number of them that are closely related to basic concepts from model theory. Examples are:

(1) Indiscernible sequences (called 1-*subsymmetric* sequences in Banach space theory);
(2) Ordinal ranks (called *ordinal indices* in analysis);
(3) Ehrenfeucht-Mostowski models (called *spreading models* in Banach space theory);
(4) Spaces of types;
(5) Stability;
(6) Ultrapowers.

In some cases, these concepts have been introduced by adapting directly a construction from model theory to the context of Banach space theory (*e.g.*, the case of Banach space ultrapowers, introduced by D. Dacunha-Castelle and J.-L. Krivine in [12]), in other cases, by analogy (*e.g.*, the case of Banach space stability, introduced by J.-L. Krivine and B. Maurey in [50]), and yet in other cases, concepts which are studied in model theory, as well as their connections with others, have been discovered independently by analysts (*e.g.*, the case of indiscernible sequences — and their construction using Ramsey's Theorem — which were introduced by

A. Brunel and L. Sucheston in the study of ergodic properties of Banach spaces; see [5]).

In addition, some concepts that play a central role in Banach space theory (*e.g.*, that of finite representability) can be seen naturally as model theoretical phenomena (a Banach space X is finitely represented in a Banach space Y if and only if Y is a model of the existential theory of X). There are even similarities between classification programs in both fields. For example, the dichotomy reflexive/unreflexive in Banach space theory is equivalent, in a categorical sense, to the dichotomy stable/unstable in model theory. (See [39].) Also, in both fields, the role played by partition theorems is regarded as fundamental.

These phenomena suggest that the relation between the two fields is rather deep. Given the remarkable technical complexity that both fields have attained in the last thirty years, we suggest that it would be desirable to have clearly understood channels of communication between them so that techniques from one field might become useful in the other. Some considerations are in order, however.

(1) First order logic is not the natural logic to analyze Banach spaces as models. Banach space theory is carried out in higher order logics, as is functional analysis in general. Furthermore, the first order theory of Banach spaces is known to be equivalent to a second order logic. (See [71].)
(2) The concepts from Banach space theory listed above are not the literal translations of their first order analogs. For instance, a Banach space ultrapower of a Banach space X is not an ultrapower of X in the sense customarily considered in model theory, and is not an elementary extension of X in the sense of first order logic. However, there is a strong analogy between the role played by Banach space ultrapowers in Banach space theory and that played by algebraic ultrapowers in model theory.

   Let us illustrate this point with a second example. What is regarded in Banach space theory as the "space of types" is not what is understood as the space of types in the first order sense. Let us recall definition given in [50]:

   > Let X be a fixed separable Banach space. A *type* on X is a function $\tau\colon X \to \mathbb{R}$ such that there exists a sequence (x_n) in X satisfying
   > $$\tau(x) = \lim_{n\to\infty} \|x + x_n\|.$$

   The space of types of X, as defined in [50], is the set of types on X with the topology of pointwise convergence. This notion of space of types is motivated by the corresponding notion from first order logic. The analogy is not entirely clear a priori. However, as we shall see, both notions of type are connected by a natural interpretation.

Model theory for structures of functional analysis. A formal framework for a model theoretical analysis of Banach spaces was first introduced by C. W. Henson in [35]; the scope expanded by Henson and the author in [36]. Although this framework was originally introduced for Banach spaces, it can be generalized naturally to include rich classes of structures from functional analysis. The unique feature of this logical approach is that, although it is appropriate for structures from functional analysis, it preserves many of the desirable characteristics of first-order model theory, *e.g.*, the compactness theorem, Löwenheim-Skolem theorems, and omitting types theorem. (In fact, it provides a natural setting for the classification theory, in the sense of [70], of structures from infinite dimensional analysis.) Furthermore, it provides a uniform foundation for the contributions mentioned above. For example, the role played by analytic ultrapowers in this framework mirrors that played by algebraic ultrapowers in first-order model theory; also, types in the sense of [50] described above correspond exactly to quantifier-free types in this context, indiscernibles in the sense of [5] are quantifier-free indiscernibles, and the kind of Banach space stability introduced in [50] corresponds exactly to quantifier-free stability of the structure.

Two famous applications. The problem of how the classical sequence spaces ℓ_p $(1 \leq p < \infty)$ and c_0 occur inside every Banach space has played a central role in Banach space geometry for more than half a century. The first example of a Banach space not containing ℓ_p or c_0 was constructed by B. S. Tsirel'son [75]. Shortly after Tsirel'son's example appeared in print, J.-L. Krivine [49] published a celebrated result (now known as Krivine's Theorem) which states that for every Banach space X there exists p with $1 \leq p \leq \infty$ such that ℓ_p is block finitely represented in X. The spectacular breakthroughs that took place in Banach space theory in the 1990's (see the historical notes at the end of the paper) confirm the long held belief that Krivine's Theorem in fact states the ultimate way in which the classical spaces ℓ_p and c_0 occur as subspaces of every Banach space.

A question that still remains open is what conditions on the norm of a Banach space guarantee that the space contains ℓ_p or c_0 almost isometrically. The most elegant partial answer to this question known so far is the theorem proved by J.-L. Krivine and B. Maurey in [50] which states that every stable Banach space contains some ℓ_p almost isometrically.

In this paper we use the model theoretical framework introduced by Henson to prove a general principle about block representability of ℓ_p in indiscernible sequences over Banach spaces (Theorem 11.1). Both Krivine's Theorem and the Krivine-Maurey theorem about ℓ_p subspaces of stable spaces follow as particular applications. In the original proofs, various concepts motivated by analogies with model theory played a fundamental role (prominently, that of Banach space ultrapower). However, these connections are in the background of the proofs and not easily visible. Here, we bring the model theoretical ideas to the foreground.

A note on the exposition. The paper is of introductory nature, and we have sacrificed generality in the interest of conciseness. Although the proofs are complete and in full detail, we have only scratched the surface of a profound field. In the brief introduction to Banach space model theory provided, we have given a definition of Banach space structure so restrictive that it excludes many important cases, such as Banach lattices or Banach algebras. The treatment of stability is absolutely minimal, and omits discussion of forking and uniform topologies on the space of types. In the part on applications, we have focused in the usefulness of indiscernibility, and have omitted altogether topics such as that of implicitly defined norms (the Banach space theoretic version of omitting types). In the historical notes at the end of the paper we give pointers to the literature on these and other central topics.

The historical notes should be regarded as an integral part of the exposition. By no means have we tried to be exhaustive. We have mentioned only the writings that have shaped the author's view of the subject.

The exposition is entirely self contained. A basic course in model theory (for example, the first three chapters of [10]) will more than suffice for the prerequisites in logic. The prerequisites in Banach space theory are minimal. We assume that the reader is familiar with the definition of the ℓ_p sequence spaces and with the definition of Banach space operator.

Finally, a word about notation. Model theorists use the letters p, q, etc. to denote types. However, in Banach space theory, these letters are reserved to denote certain parameters, specifically, the parameter p in the $L_p(\mu)$ spaces. For this reason, we have denoted types by the letters t, t', etc. We have also avoided using the letter T to denote theories, as in Banach space theory it is customarily used to denote operators.

1. Preliminaries: Banach Space Models

Banach space structures and Banach space ultrapowers. A Banach space is finite dimensional if and only if the unit ball is compact, *i.e.*, if and only if for every bounded family $(x_i)_{i\in I}$ and every ultrafilter $\mathcal{U}$ on the set I, the $\mathcal{U}$-limit

$$\lim_{i,\mathcal{U}} x_i$$

exists. If X is an infinite dimensional Banach space and $\mathcal{U}$ is an ultrafilter on a set I, there is a canonical way of expanding X to a larger Banach space $\hat{X}$ by adding for every bounded family $(x_i)_{i\in I}$ in X an element $\hat{x} \in \hat{X}$ such that $\|\hat{x}\| = \lim_{i,\mathcal{U}} \|x_i\|$. This is the construction of *Banach space ultrapower* introduced by D. Dacunha-Castelle and J.-L. Krivine in [12].

Let $(X_i)_{i\in I}$ be a family of Banach spaces. Define

$$\ell_\infty\big(\prod_{i\in I} X_i\big) = \{\ (x_i) \in \prod_{i\in I} X_i \ \mid\ \sup_{i\in I} \|x_i\| < \infty\ \}$$

$\ell_\infty(\prod_{i\in I} X_i)$ is naturally a vector space. An ultrafilter $\mathcal{U}$ on I induces a seminorm on $\ell_\infty(\prod_{i\in I} X_i)$ by defining

$$\|(x_i)\| = \lim_{i,\mathcal{U}} \|x_i\|.$$

The set $N_\mathcal{U}$ of families (x_i) in $\ell_\infty(\prod_{i\in I} X_i)$ such that $\|(x_i)\| = 0$ is obviously a closed subspace of $\ell_\infty(\prod_{i\in I} X_i)$. We define

$$\prod_{i\in I} X_i/\mathcal{U} = \ell_\infty\big(\prod_{i\in I} X_i\big)/N_\mathcal{U}.$$

The space $\prod_{i\in I} X_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultraproduct of* $(X_i)_{i\in I}$. If $X_i = X$ for every $i \in I$, the space $\prod_{i\in I} X_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultrapower of* X and is denoted $X^I/\mathcal{U}$.[1]

If (x_i) is a family in $\ell_\infty(\prod_{i\in I} X_i)$, let us denote by $(x_i)_\mathcal{U}$ the equivalence class of (x_i) in $\prod_{i\in I} X_i/\mathcal{U}$. If $X^I/\mathcal{U}$ is an ultrapower of a Banach space X, the map

[1]From a model theorist's point of view, a Banach space ultraproduct is the result of eliminating the elements of infinite norm from an ordinary ultraproduct and dividing by infinitesimals. Instead of algebraic ultrapowers, one can deal with arbitrary models of a certain theory in a first order language. However, we have chosen to use Banach space ultrapowers as, for our purposes, they provide the most straightforward approach.

$x \mapsto (x_i)_{\mathcal{U}}$, where $x_i = x$ for every $i \in I$, is an isometric embedding of X into $X^I/\mathcal{U}$. Hence, we may regard X as a subspace of $X^I/\mathcal{U}$. This embedding is generally not surjective; it is, however, when the ultrafilter $\mathcal{U}$ is principal or the space X is finite dimensional.

Suppose that T is an operator on X. Then T can be extended to an operator $T^I/\mathcal{U}$ on by defining, for $(x_i)_{\mathcal{U}}$ in $X^I/\mathcal{U}$,

$$T^I\big((x_i)_{\mathcal{U}}\big) = (\,T(x_i)\,)_{\mathcal{U}}.$$

Clearly, $\|T^I\| = \|T\|$.

If $(T_j)_{j\in J}$ is a family of operators on X and $(c_k)_{k\in K}$ is a family of elements of X, we will refer to the structure

$$\mathbf{X} = (\ X,\ T_j,\ c_k\ \mid\ j \in J,\ k \in K\ )$$

as a *Banach space structure*. The space X is called the *universe* of the structure. The structure

$$(\ X^I/\mathcal{U},\ T_j^I/\mathcal{U},\ c_k\ \mid\ j \in J,\ k \in K\ )$$

is called the $\mathcal{U}$-ultrapower of $\mathbf{X}$.

Suppose that $(\mathbf{X})_{i\in I}$ is a family of Banach space structures such that the following conditions hold:

(1) There exist sets J, K such that for each $i \in I$

$$\mathbf{X}_i = (\ X_i,\ T_{i,j},\ c_{i,k}\ \mid\ j \in J,\ k \in K\ );$$

(2) $\sup_{i\in I} \|T_{i,j}\| < \infty$ for every $j \in J$;
(3) $\sup_{i\in I} \|c_{i,k}\| < \infty$ for every $k \in K$.

Then it is natural to define for each $j \in J$ an operator $\prod_{i\in I} T_{i,j}/\mathcal{U}$ on $\prod_{i\in I} X_i/\mathcal{U}$ by letting

$$\prod_{i\in I} T_{i,j}/\mathcal{U}\left((x_i)_{\mathcal{U}}\right)_{i\in I} = (\,T_{i,j}(x_i)\,)_{\mathcal{U}}.$$

For every $j \in J$ and $k \in K$, we have

$$\|\prod_{i\in I} T_{i,j}/\mathcal{U}\| = \lim_{i,\mathcal{U}} \|T_{i,j}\|, \qquad \|((c_{i,k})_{i\in I})_{\mathcal{U}}\| = \lim_{i,\mathcal{U}} \|c_{i,k}\|.$$

The structure

$$\Big(\ \prod_{i\in I} X_i/\mathcal{U},\ \prod_{i\in I} T_{i,j}/\mathcal{U},\ (c_{i,k})_{i\in I}\ \Big|\ j \in J,\ k \in K\ \Big)$$

is called the $\mathcal{U}$-*ultraproduct of* $(\mathbf{X}_i)_{i\in I}$ and denoted

$$\prod_{i\in I} \mathbf{X}_i/\mathcal{U}.$$

If $\mathbf{X}_i = \mathbf{X}$ for every $i \in I$, the space $\prod_{i\in I} \mathbf{X}_i/\mathcal{U}$ is called the $\mathcal{U}$-*ultrapower of* $\mathbf{X}$ and is denoted $\mathbf{X}^I/\mathcal{U}$.

What is the relation between a Banach space structure and its ultrapowers? In order to answer this question we need to discuss the logic of *positive bounded formulas* and *approximate satisfaction* introduced by C. W. Henson in [34] and [35].

Positive bounded formulas. The fundamental distinction between the concept of language in Banach space model theory and the usual concept of language in first-order logic is that a Banach space language is required to come equipped with norm bounds for the constants and operators.

Suppose that X is a Banach space, C is a subset of X, and $\{ T_j \}_{j \in J}$ is a family of operators on X. Let

$$\mathbf{X} = (\ X,\ T_j,\ c \ \mid\ j \in J,\ c \in C\ ).$$

A *language* L for $\mathbf{X}$ consists of the following items.

- A binary function symbol $+$ for the vector space addition of X;
- For each rational number r, a monadic function symbol (which we denote also by r) for the scalar multiplication by r;
- For each rational number $M > 0$, monadic predicates for the sets

$$\{ x \in X \mid \|x\| \leq M \} \quad \text{and} \quad \{ x \in X \mid \|x\| \geq M \};$$

- A monadic function symbol (an *operator symbol*) for each operator T_i;
- A constant symbol for each element of C;
- Upper norm bounds for each element of C and each operator T_j.

We say that $\mathbf{X}$ is a Banach space L-structure, or simply, an L-structure. We have discussed the fact that the class of L-structures is naturally closed under ultraproducts. (Notice that the requirement that the language include bounds for each constant and operator symbols is needed here.)

The terms and formulas of L are defined as usual. The class of *positive bounded formulas* of L (or *positive bounded L-formulas*) is the class of formulas built up from the atomic formulas

$$\|t\| \leq M, \qquad \|t\| \geq M$$

(where t is a term of L and $M > 0$) by using the *positive connectives* $\wedge, \vee$ and the *bounded quantifiers*

$$\exists x(\|x\| \leq M \wedge \ \ldots\ ) \quad \text{and} \quad \forall x(\|x\| \leq M \rightarrow \ \ldots\ )$$

(where $M > 0$).

If φ is a positive bounded formula, an *approximation* of φ is a positive bounded formula that results from "relaxing" all the norm estimates in φ, as indicated by the following table.

In φ	In approximations of φ
$\|t\| \leq M$	$\|t\| \leq N \quad (N > M)$
$\|t\| \geq M$	$\|t\| \geq N \quad (N < M)$
$\exists x(\|x\| \leq M \wedge \ \ldots\ )$	$\exists x(\|x\| \leq N \wedge \ \ldots\ ) \quad (N > M)$
$\forall x(\|x\| \leq M \rightarrow \ \ldots\ )$	$\forall x(\|x\| \leq N \rightarrow \ \ldots\ ) \quad (N < M)$

1.1. **Notation.**

(1) If φ, φ' are positive bounded formulas, we write $\varphi < \varphi'$ to denote the fact that φ' is an approximation of φ.
(2) If Γ is a set of positive bounded formulas, we denote by Γ_+ the set of approximations of formulas in Γ.

The negation connective is not allowed in positive bounded formulas, nor is the implication connective, except when it occurs as part of the bounded universal quantifiers. However, for every positive bounded formula φ there is a positive bounded formula $\operatorname{neg}(\varphi)$ (the *weak negation of* φ) which in Banach space model theory plays a role analogous to that of the negation of φ. The connective neg is defined recursively as follows.

If φ is:	$\operatorname{neg}(\varphi)$ is:
$\|t\| \leq M$	$\|t\| \geq M$
$\|t\| \geq M$	$\|t\| \leq M$
$(\psi_1 \wedge \psi_2)$	$\operatorname{neg}(\psi_1) \vee \operatorname{neg}(\psi_2)$
$(\psi_1 \vee \psi_2)$	$\operatorname{neg}(\psi_1) \wedge \operatorname{neg}(\psi_2)$
$\exists x(\|x\| \leq M \wedge \psi)$	$\forall x(\|x\| \leq M \rightarrow \operatorname{neg}(\psi))$
$\forall x(\|x\| \leq M \rightarrow \psi)$	$\exists x(\|x\| \leq M \wedge \operatorname{neg}(\psi))$

1.2. **Remarks.**

(1) If φ, φ' are positive bounded formulas, then $\varphi < \varphi'$ if and only if $\operatorname{neg}(\varphi') < \operatorname{neg}(\varphi)$.

(2) If $\mathbf{X}$ is a Banach space L-structure and φ is a positive bounded L-sentence, then $\mathbf{X} \not\models \varphi$ if and only if there exists $\varphi' > \varphi$ such that $\mathbf{X} \models \operatorname{neg}(\varphi')$.

1.3. **Proposition** (Perturbation Lemma). *For every positive bounded L-formula $\varphi(x_1, \ldots, x_n)$, every $\varphi' > \varphi$, and every $M > 0$ there exists $\delta > 0$ such that for every Banach space L-structure X,*

$$X \models \bigwedge_{1\leq i\leq n} \|a_i\| \leq M \ \wedge \bigwedge_{1\leq i\leq n} \|a_i - b_i\| \leq \delta \ \wedge \ \varphi(a_1, \ldots, a_n)$$

implies

$$X \models \varphi'(b_1, \ldots, b_n).$$

Proof. By induction on the complexity of φ, using the fact that both the norm and the operations of X are uniformly continuous on every bounded subset of X (and the moduli of uniform continuity are given by the language L, so they do not depend on the structure X). ⊣

Approximate satisfaction. In order to simplify the notation, from this point on we will identify a Banach space structure with its universe.

If X is a Banach space L-structure and φ is a positive bounded L-sentence, we say that X *approximately satisfies* φ, and write

$$X \models_{\mathcal{A}} \varphi,$$

if $X \models \varphi'$ for every approximation φ' of φ.

If Γ is a set of positive bounded sentences, we say that X *approximately satisfies* Γ or that X *is a model of* Γ, and write $X \models_{\mathcal{A}} \Gamma$, if X approximately satisfies every sentence in Γ. In the notation introduced in 1.1, $X \models_{\mathcal{A}} \Gamma$ if and only if $X \models \Gamma_+$.

The notion of approximate satisfaction, rather than the usual notion of satisfaction, provides the appropriate semantics for a model theoretical analysis of Banach space structures.

The class of positive bounded formulas is not closed under negations. However, as the following proposition shows, using weak negations, it is possible to express the fact that a formula is not approximately satisfied in a structure.

1.4. **Proposition.** *If X is a Banach space L-structure and φ is a positive bounded L-sentence, then $X \not\models_{\mathcal{A}} \varphi$ if and only there exists $\varphi' > \varphi$ such that $X \models_{\mathcal{A}} \mathrm{neg}(\varphi')$.*

Proof. If $X \not\models_{\mathcal{A}} \varphi$, there exists $\varphi' > \psi$ such that $X \not\models \varphi'$. Then $X \models \mathrm{neg}(\varphi')$ and hence $X \models_{\mathcal{A}} \mathrm{neg}(\varphi')$. Assume, conversely, that there exists $\varphi' > \varphi$ such that $X \models_{\mathcal{A}} \mathrm{neg}(\varphi')$ and take sentences ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. Then $X \models \mathrm{neg}(\psi')$ (by Remark 1.2) and hence $X \not\models \psi$, so $X \not\models_{\mathcal{A}} \varphi$. ⊣

1.5. **Theorem** (Compactness). *Let Γ be a set of positive bounded L-sentences such that every finite subset of Γ is approximately satisfied by some Banach space L-structure. Then there exists a Banach space L-structure which approximately satisfies every sentence in Γ.*

Proof. Let I be the set of finite subsets of Γ_+, and for each $i \in I$ let X_i be a Banach space L-structure satisfying every sentence in i. For every finite subset Δ of Γ_+ let F_Δ be the set of all $i \in I$ such that $X_i \models \Delta$. The family $\mathcal{F}$ of sets of the form F_Δ is closed under finite intersections. If $\mathcal{U}$ is an ultrafilter on I extending $\mathcal{F}$, we have

$$\prod_{i \in I} X_i/\mathcal{U} \models_{\mathcal{A}} \Gamma.$$

⊣

A *positive bounded theory* is a set of positive bounded sentences. If X is a Banach space structure, we denote by $\mathrm{Th}_{\mathcal{A}}(X)$ the set of sentences which are approximately satisfied by X.

1.6. **Corollary.** *The following conditions are equivalent for a positive bounded theory Γ in a language L.*

(1) There exists a Banach space L-structure X such that $\Gamma = \mathrm{Th}_{\mathcal{A}}(X)$;
(2) (a) Every finite subset of Γ is approximately satisfied in some Banach space L-structure;
  (b) For every positive bounded L-sentence φ, either $\varphi \in \Gamma$ or there exists $\varphi' > \varphi$ such that $\mathrm{neg}(\varphi') \in \Gamma$.

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from Proposition 1.4. To prove (2) $\Rightarrow$ (1), use Theorem 1.5 to fix a Banach space L-structure X such that $X \models_{\mathcal{A}} \Gamma$. Then $\mathrm{Th}_{\mathcal{A}}(X) \subseteq \Gamma$, for if φ were in $\mathrm{Th}_{\mathcal{A}}(X) \setminus \Gamma$, there would exist $\varphi' > \varphi$ such that $\mathrm{neg}(\varphi') \in \Gamma \subseteq \mathrm{Th}_{\mathcal{A}}(X)$, which is impossible. Hence $\Gamma = \mathrm{Th}_{\mathcal{A}}(X)$. ⊣

If X and Y are Banach space L-structures, we say that X and Y are *approximately elementarily equivalent*, and write

$$X \equiv_{\mathcal{A}} Y,$$

if X and Y approximately satisfy the same positive bounded L-sentences. If X is a substructure of Y, we say that X is an *approximately elementary substructure of* Y, and write

$$X \prec_{\mathcal{A}} Y,$$

if $(\, X,\, a \mid a \in X\,) \equiv_{\mathcal{A}} (\, Y,\, a \mid a \in X\,)$.

1.7. **Proposition.** *Let X and Y be L-structures.*

(1) If A is a common subset of X and Y and A_0 is a dense subset of A, then

$$(\, X,\, a \mid a \in A_0\,) \equiv_{\mathcal{A}} (\, Y,\, a \mid a \in A_0\,) \quad \textit{implies} \quad (\, X,\, a \mid a \in A\,) \equiv_{\mathcal{A}} (\, Y,\, a \mid a \in A\,).$$

(2) (Tarski-Vaught Test.) *If X is an L-substructure of Y, then $X \prec_{\mathcal{A}} Y$ if and only if the following condition holds: For every positive bounded sentence φ in a language for $(\, Y,\, a \mid a \in X\,)$ of the form $\exists x(\psi(x))$ such that $Y \models_{\mathcal{A}} \varphi$ and every approximation ψ' of ψ there exists $a \in X$ such that $Y \models_{\mathcal{A}} \psi'(a)$.*

Proof. Part (1) follows from the Perturbation Lemma (Proposition 1.3). Part (2) is a straightforward induction. ⊣

1.8. **Proposition.** *Let X be a Banach space structure.*

(1) If $\hat{X}$ is an ultrapower of X, then $X \prec_{\mathcal{A}} \hat{X}$;
(2) If Y is a Banach space structure, then $Y \equiv_{\mathcal{A}} X$, if and only if there exists a Banach space structure $\hat{X} \succ_{\mathcal{A}} X$ and an embedding $f : Y \to \hat{X}$ such that $f(Y) \prec_{\mathcal{A}} \hat{X}$.

Proof. Exercise. ⊣

Recall that the *density character* of a topological space is the smallest cardinality of a dense subset of the space. For example, a space is separable if and only if its density character is $\aleph_0$.

1.9. **Proposition.** *Let X be a Banach space structure in a countable language.*

(1) (Downward Löwenheim-Skolem Theorem.) *For every set $A \subseteq X$ there exists a substructure Y of X such that $A \subseteq Y \prec_{\mathcal{A}} X$ and*

$$\text{density}(Y) = \text{density}(A).$$

(2) (Upward Löwenheim-Skolem Theorem.) *If X is infinite-dimensional, then for every cardinal κ with $\kappa \geq$ density(X) there exists an approximately elementary extension of X of density character κ.*

Proof. To prove (1), let A_0 be a dense subset of A and expand the language with constant symbols and norm bounds for the elements of A_0. Now apply Proposition 1.7 to the structure $(\, X,\, a \mid a \in A_0\,)$.

To prove (2), let X_0 be a dense subset of X and expand the language with constants symbols and norm bounds for the elements of X_0. Expand the language further with new constants symbols $\{\, c_i \,\}_{i<\kappa}$ and norm bounds $\|c_i\| = 1$ for $i < \kappa$. Every finite subset of the theory

$$\text{Th}_{\mathcal{A}}(\, X,\, a \mid a \in X_0\,) \cup \{\, \|c_i - c_j\| = 1 \;\mid\; i < j < \kappa \,\}.$$

is approximately satisfied in X, so the conclusion now follows from (1). ⊣

$(1+\epsilon)$-isomorphism and $(1+\epsilon)$-equivalence of structures. We now address the question of when two Banach spaces have isomorphic approximately elementary extensions.

In the following discussion, L_0 will denote a language that contains no operator symbols.

For every formula φ of L_0 and every rational $\epsilon > 0$ we define an approximation $\varphi_{1+\epsilon}$ as follows.

In φ	In $\varphi_{1+\epsilon}$
$\|t\| \leq M$	$\|t\| \leq M(1+\epsilon)$
$\|t\| \geq M$	$\|t\| \geq \frac{M}{1+\epsilon}$
$\psi_1 \wedge \psi_2$	$(\psi_1)_{1+\epsilon} \wedge (\psi_2)_{1+\epsilon}$
$\psi_1 \vee \psi_2$	$(\psi_1)_{1+\epsilon} \vee (\psi_2)_{1+\epsilon}$
$\exists x(\ \|x\| \leq M \wedge \psi\ )$	$\exists x(\ \|x\| \leq M(1+\epsilon) \wedge \psi_{1+\epsilon}\ )$
$\forall x(\ \|x\| \leq M \to \psi\ )$	$\forall x(\ \|x\| \leq \frac{M}{1+\epsilon} \to \psi_{1+\epsilon}\ )$

If Γ is a set of formulas of L_0, we denote by $\Gamma_{1+\epsilon}$ the set of $(1+\epsilon)$-approximations of formulas in Γ.

We say that two Banach space L_0-structures X and Y are $(1+\epsilon)$-*equivalent*, and write

$$X \equiv_{1+\epsilon} Y,$$

if for every sentence φ of L_0

$$X \models_{\mathcal{A}} \varphi \qquad \text{implies} \qquad Y \models_{\mathcal{A}} \varphi_{1+\epsilon}.$$

Let us prove that $\equiv_{1+\epsilon}$ is a symmetric relation. Suppose

$$(\,\mathrm{Th}_{\mathcal{A}}(X)\,)_{1+\epsilon} \subseteq \mathrm{Th}_{\mathcal{A}}(Y),$$

take a positive bounded sentence φ such that $Y \models_{\mathcal{A}} \varphi$, and fix $\theta > \varphi_{1+\epsilon}$ in order to show that $X \models \theta$. Choose $\varphi' > \varphi$ such that $\varphi_{1+\epsilon} < \varphi'_{1+\epsilon} < \theta$. If $X \not\models \theta$, then $X \not\models \varphi'_{1+\epsilon}$, so $X \models \mathrm{neg}(\varphi'_{1+\epsilon})$. By assumption, $Y \models_{\mathcal{A}} (\,\mathrm{neg}(\varphi'_{1+\epsilon})\,)_{1+\epsilon}$. But $(\,\mathrm{neg}(\varphi'_{1+\epsilon})\,)_{1+\epsilon}$ is equivalent to $\mathrm{neg}(\varphi')$, so $Y \models_{\mathcal{A}} \mathrm{neg}(\varphi')$. This contradicts the choice of φ, by Proposition 1.4.

If $\epsilon > 0$, two structures

$$(\ X,\ c_i\ \mid\ i \in I\ )$$

and

$$(\ Y,\ d_i\ \mid\ i \in I\ )$$

are said to be $(1+\epsilon)$-*isomorphic* if there exists a linear isomorphism $f\colon X \to Y$ such that $f(c_i) = d_i$ for every $i \in I$ and $\|f\|, \|f^{-1}\| \leq 1+\epsilon$, *i.e.*,

$$(1+\epsilon)^{-1}\|x\| \leq \|f(x)\| \leq (1+\epsilon)\|x\|$$

for every $x \in X$. The function f is called a $(1+\epsilon)$-*isomorphism.*

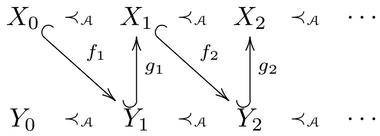
It is easy to see that two $(1+\epsilon)$-isomorphic structures are $(1+\epsilon)$-equivalent. The following is a converse of this observation.

1.10**. Theorem.** *Two Banach space L_0-structures are $(1+\epsilon)$-equivalent if and only if they have $(1+\epsilon)$-isomorphic approximately elementary extensions.*

Proof. We prove the nontrivial implication. Assume $X \equiv_{1+\epsilon} Y$. By compactness (Theorem 1.5), we construct chains of extensions

$$X = X_0 \prec_{\mathcal{A}} X_1 \prec_{\mathcal{A}} X_2 \prec_{\mathcal{A}} \cdots$$
$$Y = Y_0 \prec_{\mathcal{A}} Y_1 \prec_{\mathcal{A}} Y_2 \prec_{\mathcal{A}} \cdots$$

and embeddings



such that

$$f_n \subseteq g_n^{-1} \subseteq f_{n+1}, \qquad \text{for } n = 1, 2, \dots$$

and for every quantifier-free formula $\varphi(\bar{x})$,

$$X_n \models \varphi(\bar{a}) \qquad \text{implies} \qquad Y_{n+1} \models \varphi_{1+\epsilon}(\, f_{n+1}(\bar{a})\,)$$

and

$$Y_n \models \varphi(\bar{a}) \qquad \text{implies} \qquad X_n \models \varphi_{1+\epsilon}(\, g_n(\bar{a})\,).$$

Let $\hat{X}$ and $\hat{Y}$ denote the norm completions of $\bigcup X_n$ and $\bigcup Y_n$, respectively. Then $\hat{X} \succ_{\mathcal{A}} X$, $\hat{Y} \succ_{\mathcal{A}} Y$, and $\bigcup_{n>0} f_n$ extends to a $(1+\epsilon)$-isomorphism between $\hat{X}$ and $\hat{Y}$. ⊣

Finite representability. The notion of finite representability is the central notion in local Banach space geometry.

A Banach space X is *finitely represented* in a Banach space Y if for every finite dimensional subspace E of X and for every $\epsilon > 0$ there exists a finite dimensional subspace F of Y such that E and F are $(1+\epsilon)$-isomorphic.

If X is a Banach space structure, the *existential theory of* X, denoted $\exists\,\mathrm{Th}_{\mathcal{A}}(X)$ is the set of existential positive bounded sentences which are approximately satisfied by X.

1.11**. Proposition.** *Let X and Y be Banach spaces. The following conditions are equivalent.*

(1) X is finitely represented in Y;
(2) $\exists\,\mathrm{Th}_{\mathcal{A}}(X) \subseteq \mathrm{Th}_{\mathcal{A}}(Y)$;
(3) There exists an ultrapower of Y which contains an isometric copy of X.

Proof. The implication (3) ⇒ (1) is immediate, since an ultrapower of Y is always finitely represented in Y. The implication (1) ⇒ (2) follows from the fact that the unit ball of a finite dimensional space is compact. To prove (2) ⇒ (3), assume that X is finitely represented in Y and let Γ be set of quantifier-free diagram of X. By compactness (Theorem 1.5), there is an ultrapower $\hat{Y}$ of Y such that $\hat{Y} \models_{\mathcal{A}} \Gamma$. Since $\models_{\mathcal{A}}$ and $\models$ coincide for quantifier-free formulas, we have $\hat{Y} \models \Gamma$, so $\hat{Y}$ contains an isometric copy of X. ⊣

Types. Suppose that X is a Banach a space structure and A is a subset of X. If $\bar{c} \in X$, the *type of* $\bar{c}$ *over* A is the set

$$\mathrm{tp}(\bar{c}/A) = \{\ \varphi(\bar{x}, \bar{a}) \ \mid\ \bar{a} \in A,\ (\, X,\, a \mid a \in A\,) \models_{\mathcal{A}} \varphi(\bar{c}, \bar{a})\ \}.$$

1.12**. Proposition.** *Let X be a Banach space structure, let A be a subset of X, and let L be a language for the structure $(\, X,\, a \mid a \in A\,)$. The following conditions are equivalent for a set of positive bounded L-formulas $t(\bar{x}) = t(x_1, \dots, x_n)$.*

(1) There exists a Banach space structure $Y \succ_{\mathcal{A}} X$ and $\bar{c} \in Y$ such that $t(\bar{x}) = \mathrm{tp}(\bar{c}/A)$.

(2) (a) There exists $M > 0$ *such that the formula*

$$\bigwedge_{1\le i\le n} \|x_i\| \le M$$

is in t*;*

(b) Every L*-formula in* t_+ *is satisfied in* $(X, a \mid a \in A)$*;*

(c) For every L*-formula* $\varphi(\bar{x})$*, either* $\varphi \in t$*, or there exists* $\varphi' > \varphi$ *such that* $\operatorname{neg}(\varphi') \in t$*.*

Proof. The implication (1) ⇒ (2) is immediate from Proposition 1.4. The implication (2) ⇒ (1) follows from compactness (Theorem 1.5) and, again, Proposition 1.4. ⊣

If X is a Banach space structure, A is a subset of X, and $t(\bar{x})$ is a set of positive bounded formulas satisfying the equivalent conditions of Proposition 1.12, we say that t is a *type over* A and $\bar{c}$ *realizes* t (or $\bar{c}$ is a *realization* of t) in Y. If $\bar{x} = x_1, \dots, x_n$, we call t an n*-type*.

Fix a Banach space structure X, a subset A of X, and a language L for $(X, a \mid a \in A)$. Given a positive bounded L-formula φ, let $[\varphi]$ denote the set of types over A which contain φ. The *logic topology* is the topology on the set of types over A where the basic open neighborhoods of a type t are the sets of the form $[\varphi]$, with $\varphi \in t_+$. (These sets form a basis for a topology since t_+ is closed under finite conjunctions.) The logic topology is Hausdorff.

If $t(x_1, \dots, x_n)$ is a type and $(c_1, \dots, c_n)$ is a realization of t, we define the *norm* of t, denoted $\|t\|$, as the number $\max_{1\le i\le n} \|c_i\|$. Notice that the norm $\|t\|$ depends only on t and not on the particular realization used to compute it.

1.13. **Proposition.** *For any* $M > 0$*, the set of types of norm less than or equal to* M *is compact with respect to the logic topology.*

Proof. Fix a Banach space structure X and a subset A of X. Let $(t_i)_{i\in I}$ be a family of types over A and let $\mathcal{U}$ be an ultrafilter on I. By compactness (Theorem 1.5), for each i we can fix a Banach space structure $Y_i \succ_{\mathcal{A}} X$ such that t_i is realized in Y_i. For each $i \in I$ let $\bar{c}_i$ be a realization of t_i in Y_i. It is now easy to see that the type over A of the element of $\prod_{i\in I} Y_i/\mathcal{U}$ represented by $(\bar{c}_i)_{i\in I}$ is $\lim_{i,\mathcal{U}} t_i$. ⊣

1.14. **Remark.** It is not true that the set of types over A is compact with respect to the logic topology. Indeed, for each $n > 0$, the set $[\,\|x\| \ge n\,]$ is closed in the logic topology. The family of sets of this form has the finite intersection property. However,

$$\bigcap_{n>0} [\,\|x\| \ge n\,] = \emptyset.$$

Saturated and homogeneous structures. Let κ be an infinite cardinal. A Banach space structure X is called κ*-saturated* if every type over every subset of X of cardinality less than κ is realized in X.

The proof that every Banach space structure X has a κ-saturated approximately elementary extensions is completely analogous to the proof of the corresponding fact in first order model theory; specifically, one constructs a chain of approximately elementary extensions

$$X = X_0 \prec_{\mathcal{A}} X_1 \prec_{\mathcal{A}} \cdots \prec_{\mathcal{A}} X_i \prec_{\mathcal{A}} \cdots \qquad (i < \kappa^+) \tag{1}$$

such that whenever $i < j < \kappa^+$, every type over every subset of X_i of cardinality less than κ is realized in X_j. Then, the completion of $\bigcup_{i<\kappa^+} X_i$ is a κ-saturated approximately elementary extension of X.

Now suppose that we have a chain of structures as in (1) above such that whenever $i < j < \kappa^+$, the structure X_j is $(\mathrm{card}(X_i))^+$-saturated, and let $\hat{X} = \bigcup_{i<\kappa^+} X_i$. We say that the structure $\hat{X}$ is *κ-special*. Arguing as in Theorem 1.10, one proves that a κ-special structure $\hat{X}$ has the following property: if A is a subset of X of cardinality less than κ and $f\colon A \to X$ is such that

$$(\,\hat{X},\, a \mid a \in A\,) \equiv_{\mathcal{A}} (\,\hat{X},\, f(a) \mid a \in A\,),$$

there exists a bijection $F\colon X \to X$ extending f such that

$$(\,\hat{X},\, a \mid a \in X\,) \equiv_{\mathcal{A}} (\,\hat{X},\, F(a) \mid a \in X\,).$$

We express this fact by saying that $\hat{X}$ is *κ-homogeneous*. The argument of Theorem 1.10 also shows that if the language contains no operator symbols, then every $(1+\epsilon)$-isomorphism between two approximately elementary substructures of $\hat{X}$ of density character less than κ can be extended to a $(1+\epsilon)$-automorphism of $\hat{X}$.

The monster model. In what follows, X will denote a Banach space structure and we will regard X as being embedded as an approximately elementary substructure in a single κ-saturated, κ-special structure, where κ is a cardinal larger than any cardinal mentioned in the proofs.[2] Following the tradition (started by Shelah), we will refer to this structure as the "monster model", and denote it $\mathfrak{C}$. Our assumption on the monster model allows us to regard all the structures approximately elementary equivalent to X as substructures of $\mathfrak{C}$, and all the realizations of types over subsets of them as living inside $\mathfrak{C}$.

If $\varphi(x_1, \dots, x_n)$ is a positive bounded formula, we denote by $\varphi(\mathfrak{C})$ the subset of $\mathfrak{C}^n$ defined by φ.

If $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ are in $\mathfrak{C}$ and A is a subset of $\mathfrak{C}$, then $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$ if and only if there is an isometric automorphism f of $\mathfrak{C}$ such that $f(a_i) = b_i$ for $i = 1, \dots, n$ and f fixes A pointwise.

We follow the standard practice of identifying finite lists of elements of the monster model with finite sequences. For example, if $\bar{a} = (a_1, \dots, a_n)$ we write $\bar{a} \in \mathfrak{C}$ instead of $\bar{a} \in \mathfrak{C}^n$. Addition and scalar multiplication of finite sequences is meant to be taken componentwise.

Notice that the $\aleph_1$-saturation of the monster model implies that satisfaction and approximate satisfaction are equivalent on it.

The terms "structure", "formula", "type", and "consistent" stand, respectively, for "Banach space structure", " positive bounded formula", "positive bounded type", and "satisfied in the monster model".

2. Semidefinability of Types

Let α be an ordinal and let A be a set. A sequence $(\,\bar{a}_i \mid i < \alpha\,)$ is *indiscernible over A* if

$$\mathrm{tp}(\bar{a}_{i(0)}, \dots, \bar{a}_{i(n)}/A) = \mathrm{tp}(\bar{a}_0, \dots, \bar{a}_n/A), \qquad \text{for } i(0) < \cdots < i(n) < \alpha.$$

[2]Given that we are mostly interested in separable spaces, $\kappa = (2^{\aleph_0})^+$ will typically suffice.

2.1. **Definition.** Suppose $A \subseteq B$ and let $t(\bar{x})$ be a type over B. We say that t *splits over* A if there exist tuples $\bar{b}, \bar{c} \in B$ with $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$, a formula $\varphi(\bar{x}, \bar{y})$, and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{b}) \in t(\bar{x})$ and $\mathrm{neg}(\, \varphi'(\bar{x}, \bar{c})\,) \in t(\bar{x})$.

2.2. **Proposition.** *Suppose that* $(\, \bar{a}_i \mid i < \gamma \,)$ *is a sequence such that*

(i) $\mathrm{tp}(\bar{a}_\alpha / A \cup \{\, \bar{a}_i \mid i < \alpha \,\}) \subseteq \mathrm{tp}(\bar{a}_\beta / A \cup \{\, \bar{a}_i \mid i < \beta \,\})$ *for* $\alpha < \beta < \gamma$*;*
(ii) $\mathrm{tp}(\bar{a}_\alpha / A \cup \{\, \bar{a}_i \mid i < \alpha \,\})$ *does not split over* A *for* $\alpha < \gamma$*.*

Then the sequence $(\, \bar{a}_i \mid i < \gamma \,)$ *is indiscernible.*

Proof. We prove by induction on n that

$$\mathrm{tp}(\bar{a}_{i(0)}, \dots, \bar{a}_{i(n-1)}/A) = \mathrm{tp}(\bar{a}_0, \dots, \bar{a}_{n-1}/A), \qquad \text{for } i(0) < \cdots < i(n-1) < \gamma.$$

For $n = 1$, this is given by (i). Assume that the result is true for n and take $i(0) < \cdots < i(n) < \gamma$. By the induction hypothesis and the fact that

$$\mathrm{tp}(\bar{a}_{i(n)}/A \cup \{\, \bar{a}_i \mid i < i(n) \,\})$$

does not split over A, for every formula $\varphi(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$ with parameters in A, we have

$$\varphi(\bar{a}_{i(n)}, \bar{a}_{i(0)}, \dots, \bar{a}_{i(n-1)}) \qquad \text{if and only if} \qquad \varphi(\bar{a}_{i(n)}, \bar{a}_0, \dots, \bar{a}_{n-1}),$$

and by (i)

$$\varphi(\bar{a}_{i(n)}, \bar{a}_0, \dots, \bar{a}_{n-1}) \qquad \text{if and only if} \qquad \varphi(\bar{a}_n, \bar{a}_0, \dots, \bar{a}_{n-1}).$$

Putting together these two equivalences, we get

$$\varphi(\bar{a}_{i(n)}, \bar{a}_{i(0)}, \dots, \bar{a}_{i(n-1)}) \qquad \text{if and only if} \qquad \varphi(\bar{a}_n, \bar{a}_0, \dots, \bar{a}_{n-1}).$$

$\dashv$

2.3. **Definition.** Suppose $A \subseteq B$. A type t over B is called *semidefinable over* A if every approximation of every finite subset of t is realized in A.

2.4. **Remark.** A type t over B is semidefinable over A if and only if there exists a family $(\bar{a}_i)_{i \in I}$ in A and an ultrafilter $\mathfrak{U}$ on I such that

$$\lim_{i, \mathfrak{U}} \mathrm{tp}(\bar{a}_i / B) = t,$$

where the limit is taken in the logic topology.

2.5. **Proposition.** *Suppose that* $A \subseteq B$*. A type* t *over* B *which is semidefinable over* A *does not split over* A*.*

Proof. Suppose that $t(\bar{x})$ splits over A. Take $\bar{b}, \bar{c} \in B$ with $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$, a formula $\varphi(\bar{x}, \bar{y})$, and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{b}) \in t(\bar{x})$ and $\mathrm{neg}(\, \varphi'(\bar{x}, \bar{c})\,) \in t(\bar{x})$. Take formulas ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. Since t is semidefinable over A, there exists $\bar{a} \in A$ such that $\mathfrak{C} \models \psi(\bar{a}, \bar{b}) \wedge \mathrm{neg}(\, \psi'(\bar{a}, \bar{c})\,)$. But this contradicts the fact that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$. $\dashv$

2.6. **Proposition.** *Suppose that* $A \subseteq B \subseteq C$ *and let* $t(\bar{x})$ *be a type over* B *which is semidefinable over* A*.*

1. *(1)* *t has an extension $t'(\bar{x})$ over C which is semidefinable over A; furthermore, if $(\bar{a}_i)_{i \in I}$ is a family in A and $\mathfrak{U}$ is an ultrafilter on I such that $\lim_{i, \mathfrak{U}} \mathrm{tp}(\bar{a}_i / B) = t$, then t' can be chosen so that $\lim_{i, \mathfrak{U}} \mathrm{tp}(\bar{a}_i / C) = t'$;*
2. *(2)* *If for every $n < \omega$ every n-type over A is realized in B, then t has a unique extension $t'(\bar{x})$ over C which semidefinable over A;*

Proof. (1): We claim that if $\psi(\bar{x}) \in t$, ψ' is an approximation of ψ, and $\varphi(\bar{x}, \bar{y})$ is a formula such that

$$\{\, i \in I \mid \varphi(\bar{a}_i, \bar{c}) \,\} \notin \mathcal{U},$$

where $\bar{c} \in C$, then

$$\{\, i \in I \mid \psi'(\bar{a}_i) \wedge \operatorname{neg}(\, \varphi(\bar{a}_i, \bar{c})\,) \,\} \in \mathcal{U}.$$

Indeed, we have

$$\{\, i \in I \mid \operatorname{neg}(\, \varphi(\bar{a}_i, \bar{c})\,) \,\} \in \mathcal{U},$$

and by the hypothesis that $t = \lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/B)$ we also have

$$\{\, i \in I \mid \psi'(\bar{a}_i) \,\} \in \mathcal{U},$$

so the claim follows.

Let

$$\Gamma(\bar{x}) = \big\{\ \operatorname{neg}(\, \varphi(\bar{x}, \bar{c})\,) \ \mid\ \bar{c} \in C \text{ and } \{\, i \in I \mid \varphi(\bar{a}_i, \bar{c}) \,\} \notin \mathcal{U}\ \big\}.$$

By the claim, $t \cup \Gamma(\bar{x})$ is consistent. For every formula of the form $\varphi(\bar{x}, \bar{c})$, where $\bar{c} \in C$, we have either $\varphi \in t \cup \Gamma$ or $\operatorname{neg}(\varphi) \in t \cup \Gamma$, so $t \cup \Gamma$ is a type over C. Furthermore, if $\varphi(\bar{x}, \bar{c}) \in t \cup \Gamma$ and $\varphi' > \varphi$, we must have $\{\, i \in I \mid \varphi'(\bar{a}_i, \bar{c}) \,\} \in \mathcal{U}$. Thus, $\lim_{i,\mathcal{U}} \operatorname{tp}(\bar{a}_i/C) = t \cup \Gamma$.

(2): Suppose that $t_1(\bar{x})$ and $t_2(\bar{x})$ are distinct extensions of t over C which are semidefinable over A. Then there exist a formula $\varphi(\bar{x}, \bar{c})$ with $\bar{c} \in C$ and an approximation φ' of φ such that $\varphi(\bar{x}, \bar{c}) \in t_1$ and $\operatorname{neg}(\, \varphi'(\bar{x}, \bar{c})\,) \in t_2$. Take $\bar{b} \in B$ such that $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{c}/A)$. Take also formulas ψ, ψ' such such that $\varphi < \psi < \psi' < \varphi'$. By Proposition 2.5, t_1 does not split over A, so $\psi(\bar{x}, \bar{b}) \in t_1 \restriction B = t$; similarly, t_2 does not split over A, so $\operatorname{neg}(\, \psi'(\bar{x}, \bar{c})\,) \in t_2 \restriction B = t$. This is, of course, a contradiction. ⊣

3. Maurey Strong Types and Convolutions

3.1. **Definition.** A type t will be called a *Maurey strong type for A* if there exists a set $B \supseteq A$ such that

(1) t is over B;
(2) t is semidefinable over A;
(3) For every $n < \omega$, every n-type over A is realized in B.

In this case we say that t is a *Maurey strong type for A over B.*

The importance of Maurey strong types lies in the fact that if $t(\bar{x})$ is a Maurey strong type for A over B, then, by Proposition 2.6-(2), for every $C \supseteq B$ there exists a unique extension $t'(\bar{x})$ of $t(\bar{x})$ which is a Maurey strong type for A over C.

Suppose that $A \subseteq B, B'$, the type $t(\bar{x})$ is a strong type for A over B, and $t'(\bar{x})$ is a Maurey strong type for A over B'. We claim that if $\bar{b} = b_1, \dots, b_m \in B$ and $\bar{b}' = b_1', \dots b_m' \in B'$ are such that $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{b}'/A)$, then for every formula $\varphi(\bar{x}, y_1, \dots y_m)$ we have $\varphi(\bar{x}, \bar{b}) \in t$ if and only if $\varphi(\bar{x}, \bar{b}') \in t'$. The reason is that there is a unique Maurey strong type $t''(\bar{x})$ for A over $B \cup B'$ extending both t and t'; but then, if f is an automorphism of the monster model such that $f(t) = t'$ and f fixes A pointwise, we have $f(t'') = t''$ (since $f(t'')$ is semidefinable over A and extends both t and t'); hence, $\varphi(\bar{x}, \bar{b}) \in t$ iff $\varphi(\bar{x}, \bar{b}) \in t''$ iff $\varphi(\bar{x}, \bar{b}') \in t''$ iff $\varphi(\bar{x}, \bar{b}') \in t'$.

The preceding observation allows us to think of Maurey strong types as "types over the space of types of A". We may also choose a superset B of A such that all Maurey strong types for A under consideration are over B. (Thus, B acts as a kind of monster model for Maurey strong types for A.)

3.2. **Remark.** By Remark 2.4 and the preceding observation, $t(\bar{x})$ is a strong type for A if and only if there exist a unique extension $\mathbf{t}(\bar{x})$ of t to the monster model, a family $(\bar{a}_i)_{i\in I}$ in A, and an ultrafilter $\mathcal{U}$ on I such that

$$\mathbf{t} = \lim_{i,\mathcal{U}} \mathrm{tp}(\bar{a}_i, \mathfrak{C}),$$

where the limit is taken in the logic topology.

We now define a binary operation on Maurey strong types called the *convolution* operation.

3.3. **Proposition.** *Let $t(\bar{x})$ and $t'(\bar{x})$ be Maurey strong types for A over some $B \supseteq A$, and define a type $t * t'$ over B as follows. Let $\bar{c}$ be a realization of t, let c' be a realization of the unique extension of t' to a Maurey strong type for A over $B \cup \{\bar{c}\}$, and define*

$$t * t'(\bar{x}) = \mathrm{tp}(\bar{c} + \bar{c}'/B).$$

Then,

(1) $t * t'$ is a Maurey strong type for A;
(2) The definition of $t * t'$ is independent of the particular choice of $\bar{c}$ and $\bar{c}'$.

Proof. Let $\mathbf{t}$ be the unique extension of t to the monster model such that $\mathbf{t}$ is semidefinable over A, and similarly let $\mathbf{t}'$ be the unique extension of t' to the monster model such that $\mathbf{t}'$ is semidefinable over A. Pick families $(\bar{a}_i)_{i\in I}$ and $(\bar{a}_j)_{j\in J}$ in A and ultrafilters $\mathcal{U}, \mathcal{V}$ such that

$$\mathbf{t} = \lim_{i,\mathcal{U}} \mathrm{tp}(\bar{a}_i/\mathfrak{C})$$
$$\mathbf{t}' = \lim_{j,\mathcal{V}} \mathrm{tp}(\bar{a}_j/\mathfrak{C}).$$

Then, if $\mathbf{t} * \mathbf{t}'$ is the unique extension of $t * t'$ to the monster model such that $\mathbf{t} * \mathbf{t}'$ is semidefinable over A, we have

$$\mathbf{t} * \mathbf{t}'(\bar{x}) = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \mathrm{tp}(\bar{a}_i + \bar{a}'_j/\mathfrak{C}).$$

This shows that $t * t'$ is semidefinable over A and its definition is independent of $\bar{c}$ and $\bar{c}'$. ⊣

The proof of Proposition 3.3 provides a handy recipe to compute the convolution of two Maurey strong types; namely, if t, t' are Maurey strong types for A, $\mathbf{t}$, $\mathbf{t}'$, $\mathbf{t} * \mathbf{t}'$ are, respectively, the unique extensions of t, t', $t * t'$ to the monster model which are semidefinable over A, and

$$\mathbf{t} = \lim_{i,\mathcal{U}} \mathrm{tp}(a_i/\mathfrak{C})$$
$$\mathbf{t}' = \lim_{j,\mathcal{V}} \mathrm{tp}(a_j/\mathfrak{C}),$$

where the families $(\bar{a}_i)_{i\in I}$ and $(\bar{a}_j)_{j\in J}$ are in A and $\mathcal{U}, \mathcal{V}$ are ultrafilters on I, J respectively, then

$$\mathbf{t} * \mathbf{t}' == \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \mathrm{tp}(\bar{a}_i + \bar{a}'_j/\mathfrak{C}).$$

Furthermore, by Proposition 3.3-(1), whenever $\mathbf{t}$, $\mathbf{t}'$ $(\bar{a}_i)_{i\in I}$, $(\bar{a}_j)_{j\in J}$, $\mathcal{U}$, and $\mathcal{V}$ are as above, there exists an ultrafilter $\mathcal{W}$ in $I \times J$ such that

$$\mathbf{t} * \mathbf{t}' = \lim_{(i,j),\mathcal{W}} \operatorname{tp}(\bar{a}_i \bar{a}'_j/\mathfrak{C}).$$

An immediate consequence of these observations is the following.

3.4. **Corollary.** *The convolution operation is associative.*

4. Fundamental sequences

A scalar multiplication can be defined naturally on types naturally as follows.

4.1. **Definition.** If $t = \operatorname{tp}(\bar{a}/A)$ and r is a scalar, we denote by rt the type $\operatorname{tp}(r\bar{a}/A)$.

4.2. **Proposition.** *If t, t' are Maurey strong types and r is a scalar, then*

$$r(t * t') = (rt) * (rt');$$

Proof. Immediate from the definitions. ⊣

4.3. **Definition.** Let $t(\bar{x})$ be a Maurey strong type for A over B and let $\mathbf{t}$ be the unique extension of t to the monster model such that $\mathbf{t}$ is semidefinable over A. We will say that a sequence $(\bar{a}_n)$ is a *fundamental sequence for t* if for any choice of scalars $r_0, \dots, r_n$ we have,

$$\operatorname{tp}(r_r\bar{a}_0 + \cdots + r_n\bar{a}_n) = r_0 t * \cdots * r_n t.$$

It is immediate from this definition that a fundamental sequence for t is indiscernible over B.

Let $t(\bar{x})$ be a Maurey strong type for A over B and let $\mathbf{t}$ be the unique extension of t to the monster model which is semidefinable over A. One can produce a fundamental sequence $(\bar{a}_n)$ for t recursively by defining $\bar{a}_n$ as a realization of $\mathbf{t} \restriction B \cup \{\bar{a}_n \mid i < n\}$. Conversely, every fundamental sequence can be generated in this fashion. Thus, if $(\bar{a}_n)$ and $(\bar{a}'_n)$ are fundamental sequences for t, then there exists an automorphism of the monster model which maps $\bar{a}_n$ to $\bar{a}'_n$ and fixes B pointwise.

4.4. **Definition.** Let t be a Maurey strong type. The set of types of the form

$$r_0 t * \cdots * r_n t,$$

where $r_0, \dots, r_n$ are scalars, will be denoted $[t]$.

4.5. **Proposition.** *Let $t(\bar{x})$ be a strong type for A over B. Then there exists a type $t' \in \overline{\operatorname{span}}(t, *)$ such that $t' = -t'$.*

Proof. For every positive bounded formula $\sigma(\bar{x})$ and every rational $\epsilon \geq 0$ define a formula σ_ϵ such that:

- $\sigma_0 = \sigma$,
- $\sigma < \sigma_\epsilon < \sigma_{\epsilon'}$ if $\epsilon < \epsilon'$, and
- for every approximation σ' of σ there exists $\epsilon > 0$ such that $\sigma < \sigma_\epsilon < \sigma'$.

(One way to do this is as in Section 1.) For every positive bounded formula $\sigma(\bar{y})$ let $\mathcal{R}_\sigma$ be the real-valued function defined on the monster model by

$$\mathcal{R}_\sigma(\bar{a}) = \min\big\{\, 1, \inf\{\epsilon < 1 \mid \mathfrak{C} \models \sigma_\epsilon(\bar{a})\}\,\big\}.$$

By the Perturbation Lemma (Proposition 1.3), $\mathcal{R}_\sigma$ is uniformly continuous on every bounded subset of the monster model.

Fix a fundamental sequence $(\bar{a}_n)$ for t and a finite tuple $\bar{b}$ in B

Suppose that $\Sigma(\bar{x})$ is a finite set of formulas over $\bar{b}$, say,

$$\Sigma(\bar{x}) = \{ \sigma_1(\bar{x}, \bar{b}), \dots, \sigma_n(\bar{x}, \bar{b}) \},$$

and define a function $\mathcal{R}^\Sigma : \ell_1(n+1) \to \mathbb{R}^n$ as follows. For $(r_1, \dots, r_{n+1}) \in \ell_1(n+1)$,

$$\begin{aligned}\mathcal{R}^\Sigma(r_1, \dots, r_{n+1}) = \\ &\big(\mathcal{R}_{\sigma_1(r_1\bar{a}_1, \dots, r_{n+1}\bar{a}_{n+1}, \bar{y})}(\bar{b}) - \mathcal{R}_{\sigma_1(-r_1\bar{a}_1, \dots, -r_{n+1}\bar{a}_{n+1}, \bar{y})}(\bar{b}), \dots \\ &\dots, \mathcal{R}_{\sigma_n(r_1\bar{a}_1, \dots, r_{n+1}\bar{a}_{n+1}, \bar{y})}(\bar{b}) - \mathcal{R}_{\sigma_n(-r_1\bar{a}_1, \dots, -r_{n+1}\bar{a}_{n+1}, \bar{y})}(\bar{b}) \big).\end{aligned}$$

Notice that the map $\mathcal{R}^\Sigma$ is antipodal, *i.e.*, for $(r_1, \dots, r_n) \in \ell_1(n+1)$ we have

$$\mathcal{R}^\Sigma(-r_1, \dots, -r_n) = -\mathcal{R}^\Sigma(r_1, \dots, r_n).$$

By the Borsuk-Ulam antipodal map theorem, there exists a point $(r_1^\Sigma, \dots, r_{n+1}^\Sigma)$ in the unit sphere of $\ell_1(n+1)$ such that

$$\mathcal{R}^\Sigma(r_1^\Sigma, \dots, r_{n+1}^\Sigma) = 0.$$

This means exactly that

$$\mathrm{tp}(r_1\bar{a}_1, \dots, r_{n+1}\bar{a}_{n+1}/\bar{b}) = \mathrm{tp}(-r_1\bar{a}_1, \dots, -r_{n+1}\bar{a}_{n+1}/\bar{b}).$$

Therefore, by compactness (Theorem 1.5), there exist a type $t(\bar{x})$ over B and an ultrafilter $\mathcal{U}$ on the set of finite subsets $\Sigma(\bar{x})$ of formulas over B such that

$$\lim_{\Sigma, \mathcal{U}} \mathrm{tp}(r_1^\Sigma \bar{a}_1, \dots, r_{\mathrm{card}(\Sigma)}^\Sigma \bar{a}_{\mathrm{card}(\Sigma)} \,/\, B) = t'(\bar{x}).$$

The type t' is as desired. ⊣

4.6. **Definition.** A type t is called *symmetric* if $t = -t$.

4.7. **Remark.** Proposition 4.5 shows that symmetric types exist. Furthermore, the proof of 4.5 shows that given any Maurey strong type t, a symmetric type can be found as a limit of types of the form $r_1 t * \cdots * r_n t$, where $\sum |r_i| = 1$.

4.8. **Proposition.** *If t is a symmetric Maurey strong type over B and $(\bar{a}_n)$ is a fundamental sequence for t, then*

$$\mathrm{tp}(r_0\bar{a}_0 + \cdots + r_n\bar{a}_n) = \mathrm{tp}(\pm r_0\bar{a}_0 + \cdots + \pm r_n\bar{a}_n).$$

Proof. Immediate. ⊣

5. Quantifier-Free Types over Banach Spaces

We begin this section by establishing some notational conventions.

At this point and for the rest of the paper, we focus our attention on quantifier-free types. Thus, hereafter, the word “type” will stand for “quantifier-free type”. If $\bar{a}$ is a finite tuple and C is a subset of the monster model, $\mathrm{tp}(\bar{a}/C)$ will denote the quantifier-free type of $\bar{a}$ over C.

The type of a tuple $(a_0, \dots, a_n)$ over a set C is completely determined by the types of the elements of the linear span of $\{a_0, \dots, a_n\}$. For many purposes, this will allows us to concentrate our attention on types of elements of the monster model, rather than tuples. Thus, unless the contrary is specified, the word “type” will be used to denote quantifier-free 1-types.

If a be an element and C is a subset of the monster model, the quantifier-free type of a over C is completely determined by the formulas of the form

$$\|a+c\|,$$

where c is an element of the linear span of C. Since the function $c \mapsto \|a+c\|$ is uniformly continuous, it has a unique extension to the closed span of C. Thus, we can assume without loss of generality that C is a Banach space.

Recall that the *norm* of a 1-type is the norm of an element realizing the type.

5.1. **Remark.** If $M > 0$, the set of types of norm less than or equal to M is compact; this is in fact a restatement of the compactness theorem (Theorem 1.5), but it can be proved easily, using ultraproducts, as follows. If $(\operatorname{tp}(a_i/X))_{i\in I}$ is a family of types with $\|a_i\| \le M$ and $\mathcal{U}$ is an ultrafilter of I, then $\lim_{i,\mathcal{U}} \operatorname{tp}(a_i/X)$ is exactly the type over X realized in the $\mathcal{U}$-ultrapower of $\overline{\operatorname{span}}\{ X \cup \{a_i \mid i \in I\} \}$ by the element represented by the family $(a_i)_{i\in I}$.

The quantifier-free type of a over a Banach space X can be identified with the real-valued function

$$x \mapsto \|x+a\| \qquad (x \in X).$$

Furthermore, it is easy to see that in this identification the logic topology corresponds exactly to the product topology inherited from $\mathbb{R}^X$. The preceding remark shows that the space of types over X corresponds the closure of the set of realized types (*i.e.*, the types of the form $\operatorname{tp}(a/X)$, where $a \in X$). Thus, the density character of the space of quantifier-free types over X equals the density character of X, an in particular, the space of quantifier-free types is separable if X is separable.

5.2. **Proposition.** *Let X be a separable Banach space and let τ be a real-valued function on X. Then the following conditions are equivalent.*

(1) τ is the function corresponding to a quantifier-free 1-type over X;

(2) There exists a sequence (x_n) in X such that

$$\tau(x) = \lim_{n\to\infty} \|x_n + x\|, \qquad \text{for every } x \in X.$$

Proof. Notice that if (x_n) is as in (2), then (x_n) is bounded. Hence, (2) $\Rightarrow$ (1) follows from Remark 5.1. To prove (1) $\Rightarrow$ (2), suppose that τ corresponds to $\operatorname{tp}(c/X)$. Then let $\{ d_n \mid n \in \omega \}$ be a dense subset of X. Since every approximation formula of every formula in $\operatorname{tp}(c/X)$ is satisfied X, we can find a sequence (x_n) in X such that

$$|\, \|x_n + d_k\| - \|c + d_k\| \,| < \frac{1}{n+1}, \qquad \text{for } k = 0, \dots, n.$$

Then we have $\lim_{n\to\infty} \|x_n + x\| = \|c + x\| = \tau(x)$ for every $x \in X$. ⊣

5.3. **Definition.** Let $t(x)$ be a type over a Banach space Y. A sequence (x_n) in Y is called *approximating* for t if

$$\lim_{n\to\infty} \operatorname{tp}(x_n/Y) = t(x).$$

We also say that (x_n) *approximates* t.

5.4. **Proposition.** *Every bounded sequence in a separable Banach space X has a subsequence which approximates some type over X.*

Proof. By Remark 5.1 and Proposition 5.2. ⊣

5.5. **Proposition.** *Let X be a separable Banach space and let Y be a separable superspace of X. Then the following conditions are equivalent.*

(1) $\mathrm{tp}(a/Y)$ *is semidefinable over X;*

(2) There exists a sequence in X which approximates $\mathrm{tp}(a/Y)$.

Proof. (2) $\Rightarrow$ (1) is clear. We prove (1) $\Rightarrow$ (2). Let $\{ d_n \mid n \in \omega \}$ be a dense subset of Y. Since $\mathrm{tp}(a/Y)$ is semidefinable over X, we can find a sequence (x_n) in X such that

$$| \, \|x_n + d_k\| - \|a + d_k\| \, | < \frac{1}{n+1}, \qquad \text{for } k = 0, \dots, n.$$

Clearly, $\lim_{n\to\infty} \mathrm{tp}(x_n/Y) = \mathrm{tp}(a/Y)$. ⊣

6. Digression: Ramsey's Theorem for Analysis

In this section we discuss a form of Ramsey's Theorem which was used by A. Brunel and L. Sucheston in [5] to produce 1-subsymmetric sequences (*i.e.*, quantifier-free indiscernible sequences). The method of Brunel and Sucheston has since then become standard in Banach space geometry, and in [68] H. P. Rosenthal called it *the Ramsey principle for analysts.*

6.1. **Proposition.** *Let $(a_{m,n})_{m,n<\omega}$ be a matrix of real numbers such that* $\lim_n a_{m,n}$ *exists for every m, and*

$$\lim_m \lim_n a_{m,n} = \alpha.$$

Then there exist $k(0) < k(1) < \dots$ such that

$$\lim_{i<j} a_{k(i),k(j)} = \alpha.$$

Proof. By definition, for every $\epsilon > 0$ there exists a positive integer M_ϵ such that

$$m \geq M_\epsilon \qquad \text{implies} \qquad |\lim_n a_{m,n} - \alpha| \leq \epsilon.$$

Also, for every $\epsilon > 0$ and every fixed integer $\hat{m}$ there exists $N_\epsilon^{\hat{m}}$ such that

$$n \geq N_\epsilon^{\hat{m}} \qquad \text{implies} \qquad |a_{\hat{m},n} - \lim_n a_{\hat{m},n}| \leq \epsilon.$$

Take $k(0) < k(1) < \dots$ such that

$$k(0) \geq M_1$$
$$k(l+1) \geq \max \{ M_{2^{-l}}, N_{2^{-l}}^{k(0)}, \dots, N_{2^{-l}}^{k(l)} \}.$$

It is easy to see that

$$i < j \qquad \text{implies} \qquad |a_{k(i),k(j)} - \alpha| \leq 1/2^{i-1}.$$

⊣

We need the multidimensional version of Proposition 6.1. The proof is similar. (It can also be easily derived from Proposition 6.1 by induction and diagonalization.)

6.2. **Proposition.** *Let*

$$(\; a_{m_1,m_2,\dots,m_d} \;\; | \;\; (m_1, m_2, \dots, m_d) \in \omega^d \;)$$

be a family of real numbers such the iterated limits

$$\lim_{m_d} \dots \lim_{m_1} \; a_{m_1,m_2,\dots,m_d}$$

exist. Then there exist $k(0) < k(1) < \dots$ *such that*

$$\lim_{i_1 < i_2 < \cdots < i_d} a_{k(i_1),k(i_2),\dots,k(i_d)} = \lim_{m_d} \dots \lim_{m_1} a_{m_1,m_2,\dots,m_d}.$$

7. Spreading models

Let X be a Banach space and let Y be a superspace of X such that every quantifier-free 1-type over X is realized in Y. The proof of Proposition 2.6 shows that every quantifier-free 1-type over Y which is semidefinable over X has a unique extension over the monster model which is semidefinable over X. Thus, for the quantifier-free, 1-type context (on which we are now focusing our attention), we may define Maurey strong types for X (see Section 3) as types that are semidefinable over X and whose domain is a superspace of X where all 1-types over X are realized.

7.1. **Proposition.** *Suppose that* (x_n) *is a bounded sequence in a separable Banach space* X *and that no subsequence of* (x_n) *converges, and let* Y *be a superspace of* X *where every type over* X *is realized. Then there exists a Maurey strong type* $t(x)$ *for* X *over* Y *and a subsequence* (x_{n_k}) *of* (x_n) *such that whenever* $r_0, \dots, r_k$ *are scalars,*

$$\lim_{n_k} \dots \lim_{n_0} \operatorname{tp}(r_0 x_{n_0} + \cdots + r_k x_{n_k} / X) = (r_0 t * \cdots * r_k t) \restriction X.$$

Proof. By taking a subsequence if necessary, we can assume that (x_n) approximates a type t_0 over X. Let t be a Maurey strong type for X over Y extending t_0 over Y, let (a_n) be a fundamental sequence for t, and let $\mathbf{t}$ be the unique extension of t to the monster model such that $\mathbf{t}$ is semidefinable over X. By Proposition 5.5, there exists a subsequence (x_{n_k}) of (x_n) such that

$$\lim_k \big(x_{n_k} \,/\, X \cup \{ a_n \mid n < \omega \} \big) = \mathbf{t} \restriction (X \cup \{ a_n \mid n < \omega \}).$$

The sequence (x_{n_k}) is as required. ⊣

7.2. **Remark.** The conclusion of Proposition 7.1 says that whenever (a_n) is a fundamental sequence for t, $r_0, \dots, r_k$ are scalars, and $x \in X$,

$$\lim_{n_k} \dots \lim_{n_0} \| r_0 x_{n_0} + \cdots + r_k x_{n_k} + x \| = \| r_0 a_0 + \cdots + r_k a_k + x \|.$$

By Ramsey's Theorem (Proposition 6.2) we can assume that

$$\lim_{n_k < \cdots < n_0} \| r_0 x_{n_0} + \cdots + r_k x_{n_k} + x \| = \| r_0 a_0 + \cdots + r_k a_k + x \|.$$

In this latter case, we call the Banach space spanned by X and $\{ a_n \mid n < \omega \}$ the *spreading model* approximated by the sequence (x_{n_k}) over X. The sequence (a_n) is called the *fundamental sequence* of the spreading model. Clearly, the fundamental sequence of a spreading model over X is quantifier-free indiscernible over X. See the historical remarks for further comments on the concept of spreading model.

7.3. **Definition.** Let (x_n) be a sequence in a Banach space. We say that (y_n) is a *sequence of blocks* of (x_n) if there exist finite subsets $F_0, F_1, \dots$ of ω such that $\max F_n < \min F_{n+1}$ and $y_n \in \operatorname{span}\{ x_k \mid k \in F_n \}$ for every $n < \omega$.

7.4. **Proposition.** *Suppose that* (x_n) *is a bounded sequence in a separable Banach space* X *and that no normalized sequence of blocks of* (x_n) *converges, and let* Y *be a superspace of* X *where every type over* X *is realized. Then there there exists a* *symmetric Maurey strong type $t(x)$ for X over Y and a subsequence (x_{n_k}) of (x_n) such that whenever $r_0, \dots, r_k$ are scalars,*

$$\lim_{n_k} \dots \lim_{n_0} \operatorname{tp}(r_0 x_{n_0} + \dots + r_k x_{n_k}/X) = (r_0 t * \dots * r_k t) \upharpoonright X.$$

Proof. By Propositions 4.5 and 7.1. ⊣

Two sequences (a_n) and (b_n) are called 1-*equivalent* if the map $a_n \mapsto b_n$ determines an isometry between the span of $\{ a_n \mid n < \omega \}$ and the span of $\{ a_n \mid n < \omega \}$.

7.5. **Definition.** A sequence (a_n) in a Banach space is called 1-*unconditional* if whenever (ϵ_n) is a sequence such that $\epsilon_n = \pm 1$, the sequence $(\epsilon_n a_n)$ is 1-equivalent to (a_n).

By Proposition 4.8, every sequence which is fundamental for a symmetric Maurey strong type is indiscernible and 1-unconditional.

7.6. **Proposition.** *Suppose that (x_n) is a bounded sequence in a separable Banach space X and that no normalized sequence of blocks of (x_n) converges. Then (x_n) has a sequence of blocks which approximates a spreading model whose fundamental sequence is* 1*-unconditional.*

Proof. Immediate from Proposition 7.4 and the preceding remarks. ⊣

8. ℓ_p- and c_0-Types

8.1. **Definition.** Let $t(x)$ be a Maurey strong type. If p is a real number satisfying $1 \le p < \infty$, we will say that t is an ℓ_p-*type* if

- · t is symmetric;
- · If $r, s \ge 0$, then $rt * st = (r^p + s^p)^{1/p} t$.

The type t is called a c_0-*type* if

- · t is symmetric;
- · If $r, s \ge 0$, then $rt * st = \max(r, s) t$.

8.2. **Definition.** Let X be a Banach space and let p be a real number satisfying $1 \le p < \infty$. A sequence (a_n) is said to be *isometric over X to the standard unit basis of ℓ_p* if whenever $x \in X$ and $r_0, \dots, r_n$ are scalars,

$$\left\| x + \sum_{i=0}^{n} r_i a_i \right\| = \left\| x + \Big(\sum_{i=0}^{n} |r_i|^p \Big)^{1/p} a_0 \right\|.$$

The sequence (a_n) is said to be *isometric over X to the standard unit basis of c_0* if whenever $x \in X$ and $r_0, \dots, r_n$ are scalars,

$$\left\| x + \sum_{i=0}^{n} r_i a_i \right\| = \left\| x + \big(\max_i |r_i| \big) a_0 \right\|.$$

8.3. **Proposition.** *Suppose that X is a Banach space, Y is a superspace of X, and $t(x)$ is a symmetric strong type for X over Y. Suppose also that (a_n) is a fundamental sequence for t. Then the following conditions are equivalent for a real number $p > 0$.*

(1) $1 \le p < \infty$ and t is an ℓ_p-type;
(2) $1 \le p < \infty$ and (a_n) is isometric over Y to the standard unit basis unit of ℓ_p;

(3) For every $x \in Y$ and every natural number k,

$$\left\| x + \sum_{i=0}^{m-1} r_i a_i + (k+1)^{1/p} a_m + \sum_{i=m+1}^{n} r_i a_i \right\| = \left\| x + \sum_{i=0}^{m-1} r_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} r_i a_{i+k} \right\|.$$

Proof. (1) $\Rightarrow$ (2): We prove by induction on n that the first equality in Definition 8.2 holds. If $n \leq 1$, the equality is immediate. Assume $n \geq 1$.

Let (x_ν) be a net in X such that

$$\lim_{\nu} \operatorname{tp}(x_\nu/Y) = t.$$

Then,

$$\begin{aligned}
\left\| x + \sum_{i=0}^{n} r_i a_i \right\| &= \lim_{\nu_n} \dots \lim_{\nu_2} \left\| x + r_0 a_0 + r_1 a_1 + \sum_{i=2}^{n} r_i x_{\nu_i} \right\| \\
&= \lim_{\nu_n} \dots \lim_{\nu_2} \left\| x + (|r_0|^p + |r_1|^p)^{1/p} a_0 + \sum_{i=2}^{n} r_i x_{\nu_i} \right\| \\
&= \left\| x + (|r_0|^p + |r_1|^p)^{1/p} a_0 + \sum_{i=2}^{n} r_i a_i \right\| \\
&= \lim_{\nu_0} \left\| x + (|r_0|^p + |r_1|^p)^{1/p} x_{\nu_0} + \sum_{i=2}^{n} r_i a_i \right\| \\
&= \lim_{\nu_0} \left\| x + (|r_0|^p + |r_1|^p)^{1/p} x_{\nu_0} + \big(\sum_{i=2}^{n} |r_i|^p\big)^{1/p} a_n \right\| \\
&= \left\| x + (|r_0|^p + |r_1|^p)^{1/p} a_0 + \big(\sum_{i=2}^{n} |r_i|^p\big)^{1/p} a_n \right\| \\
&= \left\| x + \big(\sum_{i=0}^{n} |r_i|^p\big)^{1/p} a_0 \right\|.
\end{aligned}$$

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are immediate. We prove (3) $\Rightarrow$ (2).

Fix scalars $r_0, \dots, r_n$. Since t is symmetric, we can also assume $r_0, \dots, r_n \geq 0$. Furthermore, by a density argument, we may assume without loss of generality that r_i^p is rational, for $i = 0, \dots, n$. We can therefore fix a positive integer M such that $M r_i^p$ is an integer, for $i = 0, \dots, n$. By the indiscernibility of (a_n) over Y, for every $x \in Y$ we have

$$\begin{aligned}
\left\| M^{1/p} x + \sum_{i=0}^{n} (M r_i^p)^{1/p} a_0 \right\| &= \left\| M^{1/p} x + \sum_{i=0}^{n} \sum_{j=0}^{M r_i^p - 1} a_{i+j} \right\| \\
&= \left\| M^{1/p} x + \big(\sum_{i=0}^{n} M r_i^p\big)^{1/p} a_i \right\|.
\end{aligned}$$

Dividing by $M^{1/p}$, we obtain the desired result. ⊣

8.4. **Proposition.** *Suppose that X is a Banach space, Y is a superspace of X, and $t(x)$ is a symmetric strong type for X over Y. Suppose also that (a_n) is a fundamental sequence for t. Then the following conditions are equivalent.*

(1) *t is a c_0-type;*
(2) *(a_n) is isometric over Y to the standard unit basis of c_0;*
(3) *For every $x \in Y$ and every natural number k,*

$$\left\| x + \sum_{i=0}^{m-1} r_i a_i + a_m + \sum_{i=m+1}^{n} r_i a_i \right\| = \left\| x + \sum_{i=0}^{m-1} r_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} r_i a_{i+k} \right\|.$$

Proof. Similar to the proof of Proposition 8.3 ⊣

8.5. **Remark.** The equivalence (2) ⇔ (3) in Propositions 8.3 and 8.4 holds for arbitrary (a_n). (The assumption that (a_n) is fundamental is not needed in the proof.)

9. Extensions of Operators by Ultrapowers

In this section we prove a simple but powerful observation about ultrapowers of operators, namely, Proposition 9.3. This proposition will be used in Section 11 to transform indiscernible sequences. In this section, all Banach spaces mentioned are assumed to be complex.

Recall that the set of operators on a Banach space is a Banach space, with the norm of an operator T defined by $\sup_{\|x\|\le 1} \|T(x)\|$. The identity operator is denoted I. Note that if T, W are operators on X, then $\|TW\| \le \|T\|\,\|W\|$.

9.1. **Proposition.** *Let X be a Banach space.*

(1) *If T is an operator on X with $\|T\| < 1$, then $I - T$ is invertible.*
(2) *The set of invertible operators on X is open in the norm topology.*

Proof. (1): Let $W = \sum_n T^n$. It is easy to see that W is an operator on X and $(I-T)W = W(I-T) = I$.

(2): Suppose that W is an invertible operator on X. If T is any other operator, $\|I - TW^{-1}\| \le \|W - T\|\,\|W^{-1}\|$. Thus, if $\|W - T\| < \|W^{-1}\|^{-1}$, then TW^{-1} is invertible by (1), and hence so is T. ⊣

The *spectrum* of an operator T on a complex Banach space is

$$\{\, \lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible} \,\}.$$

It follows from Proposition 9.1 that the spectrum of an operator is a closed subset of $\mathbb{C}$.

9.2. **Proposition.** *Let T be an operator on a complex Banach space X and let λ be an element of the boundary of the spectrum of T. Then there exists an ultrapower $(\hat{X}, \hat{T})$ of (X, T) and $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}(e) = \lambda e$.*

Proof. By replacing T with $T - \lambda I$, we can assume that $\lambda = 0$. Note that then 0 is in the spectrum of T, since it is in the boundary and the spectrum is closed.

Suppose that the conclusion of the proposition is false. Then there exists $\delta > 0$ such that $\inf_{\|x\|=1} \|T(x)\| \ge \delta$. Also, since 0 is in the boundary of the spectrum of T, we can find complex numbers μ of arbitrarily small modulus such that $T - \mu I$

is invertible. Fix such μ with $|\mu| < \frac{\delta}{2}$. Then, by Proposition 9.1, the operator $1 + \mu(T - \mu I)^{-1}$ is invertible. But then so is

$$(T - \mu I)(1 + \mu(T - \mu I)^{-1}) = T,$$

which contradicts the fact that 0 is in the spectrum of T. ⊣

9.3. **Proposition.** *Let $(T_i \mid i \in I)$ be a family of operators on a complex Banach space X such that $T_i T_j = T_j T_i$ for $i, j \in I$, and suppose that $(\lambda_i \mid i \in I)$ is a family of complex numbers such that λ_i is in the boundary of the spectrum of T_i, for every $i \in I$. Then there exist*

- *an ultrapower $(\hat{X}, \hat{T}_i \mid i \in I)$ of $(X, T_i \mid i \in I)$, and*
- *an element $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}_i(e) = \lambda_i e$ for every $i \in I$.*

Proof. By compactness, it suffices to consider the case when I is finite. We prove the proposition by induction on the number of elements of I. If I is a singleton, our proposition is just Proposition 9.2. Assume, then, that $I = \{1, \dots, n\}$.

By induction hypothesis, there exists an ultrapower $(\hat{X}, \hat{T}_i \mid i \le n)$ of $(X, T_i \mid i \le n)$, a family of complex numbers $(\lambda_i \mid i \le n)$, and $e \in \hat{X}$ with $\|e\| = 1$ such that $\hat{T}_i(e) = \lambda_i e$ for $i < n$. Let

$$Y = \{ x \in \hat{X} \mid \hat{T}_i(x) = \lambda_i x \text{ for } i < n \}.$$

Since $\hat{T}_n$ commutes with $\hat{T}_i$ for $i < n$, we have $\hat{T}_n(Y) \subseteq Y$. For $i \le n$, let $U_i : Y \to Y$ be the restriction of $\hat{T}_i$ to Y, and consider the structure

$$\mathbf{Y} = (Y, U_i \mid i \le n).$$

Proposition 9.2, provides an ultrapower $(\hat{Y}, \hat{U}_i \mid i \le n)$ of $\mathbf{Y}$ and an element $f \in \hat{Y}$ with $\|f\| = 1$ satisfying $\hat{U}_i(f) = \lambda_i f$ for $i = 1, \dots, n$. By compactness, $(\hat{Y}, \hat{U}_i \mid i \le n)$ can be embedded in an ultrapower of $(\hat{X}, \hat{T}_i \mid i \le n)$, so the proposition follows. ⊣

10. Where Does the Number p Come From?

Our goal in the next sections will be to find ℓ_p-like spaces inside Banach spaces. A common question is: how does the p arise? Generally, p is given by a variation of the following elementary observation.

10.1. **Proposition.** *Let $(\lambda_n)_{n \ge 1}$ be a sequence of real numbers such that*

(1) $1 = \lambda_1 \le \lambda_2 \le \dots$

(2) $\lambda_m \lambda_n = \lambda_{mn}$.

Then, either $\lambda_n = 1$ for every n, or there exists a number $p > 0$ such that $\lambda_n = n^{1/p}$ for every n.

Proof. Suppose $\lambda_2 > 1$ and let $p = \frac{\log 2}{\log(\lambda_2)}$. Fix integers $m, n \ge 2$. For every integer k there exists an integer $h = h(k)$ such that $m^{h(k)} \le n^k < m^{h(k)+1}$. By (1) and (2), we have $\lambda_m^{h(k)} \le \lambda_n^k \le \lambda_m^{h(k)+1}$. Hence,

$$\left| \, k \frac{\log n}{\log m} - k \frac{\log \lambda_n}{\log \lambda_m} \, \right| \le 1.$$

By letting $k \to \infty$, we obtain

$$\frac{\log \lambda_n}{\log n} = \frac{\log \lambda_m}{\log m}.$$

Hence, $\lambda_n = n^{1/p}$. $\dashv$

11. Block Representability of ℓ_p in Types

If t is a strong type, $u \in [t]$ and $u = s_0 t * \cdots * s_m t$, we denote by $r_0 u * \cdots * r_k u$ the element of $[t]$ given by

$$r_0 s_0 t * \cdots * r_0 s_m t * \cdots * r_k s_0 t * \cdots * r_k s_m t.$$

11.1. **Theorem.** *Let t be a nonzero symmetric Maurey strong type for X. Then there exists a sequence (e_n) with the following properties.*

(1) (e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \le p < \infty$;

(2) There exists a sequence of types (u_l) in $[t]$ such that for scalars $r_0, \dots, r_k$,

$$\operatorname{tp}(\, r_0 e_0 + \cdots + r_k e_k \,/\, X\,) = \lim_l \, (r_0 u_l * \cdots * r_k u_l) \restriction X.$$

In the proof of Theorem 11.1, we will use Banach space operators and refer to Section 9. Since the spectrum of an operator is guaranteed to be nonempty only when the field of scalars is the field of complex numbers, we will use the concept of *complexification* of a Banach space which we explain below.

Let t be a nonzero symmetric symmetric Maurey strong type for X over Y. Suppose that $(\Sigma, <)$ is an ordered set and $(a_\nu)_{\nu \in \Sigma}$ is a family such that, for scalars $r_0, \dots, r_k$,

$$\operatorname{tp}(\, r_0 a_{\nu_0} + \cdots + r_k a_{\nu_k} \,/\, Y\,) = r_0 t * \cdots * r_k t, \qquad \text{if } \nu_0 < \cdots < \nu_k \text{ are in } \Sigma$$

(so (a_ν) is necessarily indiscernible over X). Let $Z = \overline{\operatorname{span}}\{\, a_\nu \mid \nu \in \Sigma\,\}$. Then Z can be extended to a complex Banach space naturally by defining, for $r_0, \dots, r_k \in \mathbb{C}$,

$$\|r_0 a_{\nu_0} + \cdots + r_k a_{\nu_k}\| = \||r_0| a_{\nu_0} + \cdots + |r_k| a_{\nu_k}\|.$$

The resulting complex Banach space is called the *complexification* of Z and is denoted $Z^{\mathbb{C}}$. Since t is symmetric, the norm of $Z^{\mathbb{C}}$ extends that of Z. If $z = \sum r_i a_{\nu_i} \in Z^{\mathbb{C}}$, the element $\sum |r_i| a_{\nu_i} \in Z$ is denoted $|z|$ and called the *modulus* of z.

Proof of Theorem 11.1. Let $(a_q)_{q \in \mathbb{Q} \cap (0,1)}$ be an indiscernible family such that for scalars $r_0, \dots, r_k$,

$$\operatorname{tp}(\, r_0 a_{q_0} + \cdots + r_k a_{q_k} \,/\, X\,) = r_0 t * \cdots * r_k t, \qquad \text{if } q_1 < \cdots < q_k.$$

Let $Z = \overline{\operatorname{span}}\{\, a_i \mid i \in I\,\}$. For each positive integer n define an operator $T_n \colon Z^{\mathbb{C}} \to Z^{\mathbb{C}}$ as follows. If $q_0 < \cdots < q_k$ are in $\mathbb{Q} \cap (0,1)$,

$$T_n\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big) = \sum_{j=0}^{n-1} \sum_{i=0}^{k} r_i \, a_{\frac{q_i}{n} + \frac{j}{n}}.$$

We show that for every m, n,

(i) $T_n(z) \le n\|z\|$, for $z \in Z^{\mathbb{C}}$;

(ii) $T_m \circ T_n = T_{mn}$;

(iii) $\|T_n(z)\| \le \|T_{n+1}(z)\|$, for $z \in Z^{\mathbb{C}}$.

Properties (i) and (ii) follow from the indiscernibility of (a_q). To prove (iii), notice that since t is symmetric (and (a_q) is indiscernible), for $q_0 < \cdots < q_{k+1}$ in $\mathbb{Q} \cap (0,1)$ we have

$$
\begin{aligned}
2\Big\| T_n\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\|
&= \Big\| \sum_{j=0}^{n}\sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n+1}+\frac{j}{n+1}} + \sum_{j=0}^{n-1}\sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n+1}+\frac{j}{n+1}} - \sum_{i=0}^{k} r_i\, a_{\frac{q_i}{n+1}+\frac{n}{n+1}} \Big\| \\
&\le \Big\| T_{n+1}\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\| + \Big\| T_{n+1}\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\| \\
&= 2\Big\| T_{n+1}\Big(\sum_{i=0}^{k} r_i a_{q_i}\Big)\Big\|.
\end{aligned}
$$

Now we apply Proposition 9.3 to find an extension $(\hat{Z}, \hat{T}_n \mid n \geq 1,)$ of $(Z^{\mathbb{C}}, T_n \mid n \geq 1)$, a sequence (λ_n) of complex numbers, and a nonzero element $e \in \hat{Z}$ such that $\hat{T}_n(e) = \lambda_n e$. We argue below that λ_n can be taken in $\mathbb{R}$, and furthermore, positive.

By the definition of modulus in $Z^{\mathbb{C}}$ for $z \in Z^{\mathbb{C}}$ we have

$$\|T_n(|z|) - |\lambda_n||z|\| \leq \|T_n(z) - \lambda_n z\|.$$

Hence, the same inequality remains true if $Z^{\mathbb{C}}$ is replaced by $\hat{Z}$ and T_n by $\hat{T}_n$. Therefore λ_n can be replaced by $|\lambda_n|$. From now on, we forget about the complexification of Z and switch our attention back to Z.

By Proposition 10.1 and (ii)–(iii) above, we conclude that either $\lambda_n = 1$ for every n, or there exists a real number $p > 0$ such that $\lambda_n = n^{1/p}$.

Let (z_l) be a sequence in the span of (a_q) such that $\lim_l \operatorname{tp}(z_l/X) = \operatorname{tp}(e/X)$ and $\lim_l T_n(z_l) = \lambda_n z_l$, and let $u_l \in [t]$ be such that $\operatorname{tp}(z_l/X) = u_l$.

Let $\{ c_n \mid n < \omega \}$ be a set of new constants and let $\Gamma(c_n)_{n<\omega}$ be a set of sentences expressing the following facts:

(v) $\operatorname{tp}(r_0 c_0 + \cdots + r_k c_k \,/\, X) = \lim_l (r_0 u_l * \cdots * r_k u_l) \restriction X$ for any scalars $r_0, \dots, r_k$;
(vi) If $x \in X$ and $r_0, \dots, r_n$ are scalars,

$$\Big\| x + \sum_{i=0}^{m-1} r_i c_i + \lambda_{k+1} c_m + \sum_{i=m+1}^{n} r_i c_i \Big\| = \Big\| x + \sum_{i=0}^{m-1} r_i c_i + \sum_{i=m}^{m+k} c_i + \sum_{i=m+1}^{n} r_i c_{i+k} \Big\|.$$

Every finite finite subset of $\Gamma(c_n)_{n<\omega}$ is realized in Z by interpreting the constants with z_l for sufficiently large l. Let $(e_n)_{n<\omega}$ realize $\Gamma(c_n)_{n<\omega}$. By Remark 8.5, if $\lambda_n = n^{1/p}$, then $1 \leq p < \infty$ and (e_n) is isometric over X to the standard unit basis of ℓ_p; otherwise $\lambda_n = 1$ for every n and (e_n) is isometric to c_0 over X. ⊣

12. Krivine's Theorem

If $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are finite sequences, X is a Banach space, and $\epsilon > 0$, we write

$$\operatorname{tp}(a_0, \dots, a_k \,/\, X) \overset{1+\epsilon}{\sim} \operatorname{tp}(b_n, \dots, b_k \,/\, X)$$

and say that the types $\operatorname{tp}(a_0, \dots, a_k \,/\, X)$ and $\operatorname{tp}(b_0, \dots, b_k \,/\, X)$ $(1+\epsilon)$-*equivalent over* X if there is a $(1+\epsilon)$-isomorphism f from $\overline{\operatorname{span}}\big\{ \{ a_i \mid i \leq k \} \cup X \big\}$ onto

$\overline{\text{span}}\{\,\{\,b_i \mid i \leq k\,\} \cup X\,\}$ such that $f(a_i) = b_i$ for $i = 1, \dots, k$ and f fixes X pointwise.

12.1. **Proposition.** *Suppose (a_n) is a fundamental sequence for a nonzero symmetric Maurey strong type for a Banach space X. Then there exists a sequence (e_n) such that*

(1) (e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \leq p < \infty$;
(2) For every $\epsilon > 0$ and every $k \in \omega$ there exist blocks $b_0, \dots, b_k$ of (a_n) satisfying

$$\operatorname{tp}(\,e_0, \dots, e_k \,/\, X\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,b_0, \dots, b_k \,/\, X\,).$$

Proof. Suppose (a_n) is fundamental for a symmetric Maurey strong type t for X. By Theorem 11.1 there exists a sequence (e_n) such that

(1) (e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \leq p < \infty$.
(2) There exists a sequence of types (u_l) in $[t]$ such that for scalars $r_0, \dots, r_k$,

$$\operatorname{tp}(\,r_0e_0 + \dots + r_ke_k \,/\, X\,) = \lim_l\,(r_0u_l * \dots * r_ku_l) \restriction X.$$

Fix $\epsilon > 0$ and $k \in \omega$. By (2-b) above and the fact that the unit ball of $(\mathbb{R}^k, \|\;\|_\infty)$ is compact, we find blocks $b_0, \dots, b_k$ of (a_n) such that whenever $r_0, \dots, r_k$ are scalars,

$$\operatorname{tp}(\,r_0e_0 + \dots + r_ke_k \,/\, X\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,r_0b_0 + \dots + r_kb_k \,/\, X\,).$$

The conclusion of the proposition now follows. ⊣

A sequence (e_n) is *block finitely represented* in a sequence (a_n) if for every $\epsilon > 0$ and every $k < \omega$ there exist blocks $e_0, \dots, e_k$ of (a_n) such that

$$\operatorname{tp}(\,e_0, \dots, e_k \,/\, \emptyset\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,b_0, \dots, b_k \,/\, \emptyset\,).$$

12.2. **Theorem** (Krivine's Theorem). *Let (x_n) be a bounded sequence in a Banach space such that no normalized sequence of blocks of (x_n) converges. Then, either there exists p with $1 \leq p < \infty$ such that ℓ_p is block finitely represented in (x_n), or c_0 is block finitely represented in (x_n).*

Proof. After replacing (x_n) with a sequence of blocks of it if necessary, Proposition 7.4 allows us to fix a symmetric Maurey strong type $t(x)$ for X such that whenever $r_0, \dots, r_k$ are scalars,

$$\lim_{n_k < \dots < n_0} \operatorname{tp}(r_0x_{n_0} + \dots + r_kx_{n_k}/X) = (r_0t * \dots * r_kt) \restriction X.$$

Let (a_n) be a fundamental sequence for t. Then, whenever $r_0, \dots, r_k$ are scalars,

$$(\dagger) \qquad \lim_{n_k < \dots < n_0} \operatorname{tp}(\,r_0x_{n_0} + \dots + r_kx_{n_k} \,/\, X\,) = \operatorname{tp}(\,r_0a_0 + \dots + r_ka_k \,/\, X\,)$$

Fix $\epsilon > 0$ and $k < \omega$, and by Proposition 12.1, let (e_n) be such that

(1) (e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \leq p < \infty$;
(2) There exist blocks $b_0, \dots, b_k$ of (a_n) with

$$(\ddagger) \qquad \operatorname{tp}(\,e_0, \dots, e_k \,/\, X\,) \overset{1+\epsilon}{\sim} \operatorname{tp}(\,b_0, \dots, b_k \,/\, X\,).$$

By (†), we find blocks $y_0, \dots, y_k$ of (x_n) such that

$$\mathrm{tp}(\, y_0, \dots, y_k \,/\, X\,) \overset{1+\epsilon}{\sim} \mathrm{tp}(\, b_0, \dots, b_k \,/\, X\,).$$

Putting this together with (‡), we obtain

$$\mathrm{tp}(\, y_0, \dots, y_k \,/\, X\,) \overset{(1+\epsilon)^2}{\sim} \mathrm{tp}(\, e_0, \dots, e_k \,/\, X\,),$$

and Krivine's Theorem follows since ϵ is arbitrary. ⊣

13. Stable Banach Spaces

A separable Banach space X is *stable* if whenever (x_m) and (y_n) are bounded sequences in X and $\mathcal{U}, \mathcal{V}$ are ultrafilters on $\mathbb{N}$,

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\|.$$

Let $\varphi(\bar{x}, \bar{y})$ be a positive bounded formula and let $\varphi'(\bar{x}, \bar{y})$ be an approximation of φ (see Section 1). We will say that the pair φ, φ' *has the order property* in the space X if there exist bounded sequences $(\bar{x}_m)$ and $(\bar{y}_n)$ in X such that

$$\begin{aligned} X &\models \varphi(\bar{x}_m, \bar{y}_n), && \text{if } m \le n; \\ X &\models \mathrm{neg}(\, \varphi'(\bar{x}_m, \bar{y}_n)\,), && \text{if } m > n. \end{aligned}$$

13.1. **Proposition.** *A separable Banach space X is stable if and only if no pair of quantifier-free positive bounded formulas has the order property in X.*

Proof. Every quantifier-free positive formula $\varphi(\bar{x}, \bar{y})$ is equivalent to a conjunction of disjunctions of formulas of the form

$$\|\Lambda(\bar{x}, \bar{y})\| \le r \qquad \text{or} \qquad \|\Lambda(\bar{x}, \bar{y})\| \ge r,$$

where r is a scalar and $\Lambda(\bar{x}, \bar{y})$ is a linear combination of $\bar{x}$ and $\bar{y}$. Hence, by the pigeonhole principle, a pair of quantifier-free formulas has the order property in X if and only if there exist bounded sequences (x_m) and (y_n) in X such that

$$\sup_{m<n} \big(\|x_m + y_n\|\big) \neq \inf_{m>n} \big(\|x_m + y_n\|\big).$$

But, by Ramsey's Theorem (Proposition 6.1), this is equivalent to saying that X is unstable. ⊣

Suppose that (x_m) and (x'_m) are bounded sequences in X and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ such that

$$\lim_{m,\mathcal{U}} \mathrm{tp}(x_m/X) = \lim_{m,\mathcal{U}} \mathrm{tp}(x'_m/X).$$

Then, if (y_m) is a bounded sequence in X and $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$,

$$\lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x_m + y_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|x'_m + y_n\|.$$

Similarly, if (y_n) and (y'_n) are bounded sequences in X and $\mathcal{V}$ is an ultrafilter on $\mathbb{N}$ such that

$$\lim_{n,\mathcal{V}} \mathrm{tp}(y_n/X) = \lim_{n,\mathcal{V}} \mathrm{tp}(y'_n/X),$$

then, whenever (x_m) is a bounded sequence in X and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, we have

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y_n\| = \lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|x_m + y'_n\|.$$

Thus, if X is stable, we can define a binary operation $*$ on the space of types over X as follows. Let t, t' be types over X and let (x_m) and (y_n) be sequences in X such that $t = \lim_{m,\mathcal{U}} \mathrm{tp}(x_m/X)$ and $t' = \lim_{n,\mathcal{V}} \mathrm{tp}(y_n/X)$. We define

$$t * t' \;=\; \lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \mathrm{tp}(x_m + y_n/X).$$

The preceding remarks prove that this operation is well defined. This operation is called the *convolution* on the space of types of X. Notice that there is no conflict between this use of the word "convolution" and the general concept of convolution introduced in Section 3.

13.2**. Proposition.** *The convolution on the space of types of a stable Banach space is commutative and separately continuous.*

Proof. Immediate from the definitions. ⊣

13.3**. Remark.** A space X is stable if and only if there exists a separately continuous binary operation $*$ on the space of types over X which extends the addition of X in the sense that if $x, y \in X$,

$$\mathrm{tp}(x/X) * \mathrm{tp}(y/X) = \mathrm{tp}(x + y/X).$$

Examples of stable Banach spaces include the ℓ_p and L_p spaces. For a proof that these spaces are stable, we refer the reader to [50]. For further examples of stable spaces, see [18, 60, 61].

13.4**. Remark.** The space c_0 is not stable. For each $n < \omega$ let x_n be the nth vector of the standard unit basis of c_0, and let $y_n = x_0 + \cdots + x_n$. Then

$$\|x_n + y_m\| = \begin{cases} 1, & \text{if } m > n \\ 2, & \text{if } m \le n. \end{cases}$$

Since the property of being stable is closed under subspaces, no stable space can contain c_0.

14. Block Representability of ℓ_p in Types Over Stable Spaces

14.1**. Definition.** Let t be a symmetric type over X and let $1 \le p < \infty$. We will say that ℓ_p (or ℓ_∞) is *block represented in* $[t]$ if there exists a sequence (e_n) such that

(1) (e_n) is isometric over X to the standard unit basis of ℓ_p (respectively, c_0);
(2) There exists a sequence of types (u_l) in $[t]$ such that for scalars $r_0, \dots, r_k$,

$$\mathrm{tp}(\, r_0 e_0 + \cdots + r_k e_k \,/\, X\,) = \lim_l\, (r_0 u_l * \cdots * r_k u_l).$$

For a symmetric type t over X, we define

$$\mathfrak{p}[t] = \{\; p \in [1, \infty] \;\mid\; \ell_p \text{ is block represented in } [t] \;\}$$

Theorem 11.1 says exactly that for every Banach space X and every nonzero symmetric type t over X, the set $\mathfrak{p}[t]$ is nonempty.

14.2**. Proposition.** *Suppose that X is stable. If t, t' are symmetric types over X such that $t \in \overline{[t']}$, then $\mathfrak{p}[t] \subseteq \mathfrak{p}[t']$.*

Proof. Suppose that $p \in \mathfrak{p}[t]$ and take (e_n), and (u_l) corresponding to p and $[t]$ as in Theorem 11.1. Since $u_l \in [t]$, we can write

$$u_l = s_0^l t * \cdots * s_{j(l)}^l t,$$

where $s_0^l, \ldots, s_{j(l)}^l$ are scalars. Also, since $t \in \overline{[t']}$, there exists a sequence (w_m) in $[t']$ such that $t = \lim_m w_m$. Then for any scalars $r_1, \ldots, r_k$ we have the following equalities. The last one follows from the separate continuity of the convolution.

$$\begin{aligned}
&\mathrm{tp}(\, r_0 e_0 + \cdots + r_n e_k \,/\, X\,)\\
&= \lim_l \Big[\, r_0(s_0^l t * \cdots * s_{j(l)}^l t) * \cdots * r_k(s_0^l t * \cdots * s_{j(l)}^l t) \,\Big]\\
&= \lim_l \Big[\, r_0\big(s_0^l \lim_m w_m * \cdots * s_{j(l)}^l \lim_m w_m\big) * \cdots * r_k\big(s_0^l \lim_m w_m * \cdots * s_{j(l)}^l \lim_m w_m\big) \,\Big]\\
&= \lim_l \Big[\, r_0 \lim_{m_0} \ldots \lim_{m_{j(l)}} (s_0^l w_{m_0} * \cdots * s_{j(l)}^l w_{m_{j(l)}}) * \ldots\\
&\qquad\qquad \cdots * r_k \lim_{m_0} \ldots \lim_{m_{j(l)}} (s_0^l w_{m_0} * \cdots * s_{j(l)}^l w_{m_{j(l)}}) \Big].
\end{aligned}$$

Now Ramsey's Theorem (Proposition 6.2) allows us to replace each of the iterated limits inside the square brackets by the same single limit. These limits can be taken out of the square brackets by the separate continuity of the convolution. Thus, by Ramsey's Theorem, we conclude $p \in \mathfrak{p}[t']$. ⊣

14.3. **Proposition.** *Suppose that X is stable. Then there exists a type t over X such that*

(1) t is symmetric;
(2) $\|t\| = 1$;
(3) $\mathfrak{p}[t'] = \mathfrak{p}[t]$ for every type $t' \in \overline{[t]}$ of norm 1.

Proof. Suppose that the conclusion of the proposition is false. We construct, inductively, a sequence $(\, t_i\,)_{i<(2^{\aleph_0})^+}$ of types over X such that

(1) t_i is symmetric;
(2) $\|t_i\| = 1$;
(3) $t_i \in \overline{[t_j]}$ for $i > j$;
(4) $\mathfrak{p}[t_i] \subsetneq \mathfrak{p}[t_j]$ for $i > j$.

This is clearly impossible.

We construct t_i by induction on i. The case when i is a successor ordinal is given by assumption. Suppose that i is a limit ordinal. Fix an ultrafilter $\mathcal{U}$ on i. By compactness, there exists a type t' over X such that $\lim_{j<i,\mathcal{U}} t_j = t'$. Conditions (1)–(3) are satisfied by letting $t_i = t'$.

⊣

15. ℓ_p-Subspaces of Stable Banach Spaces

Let $(\Sigma, \leq)$ be a partially ordered set. For an ordinal α we define the set Σ^α as follows.

- $\Sigma^0 = \Sigma$;
- If $\alpha = \beta + 1$,

$$\Sigma^{\alpha+1} = \{\ \xi \in \Sigma^\alpha \ \mid \ \text{There exists } \eta \in \Sigma^\alpha \text{ with } \eta > \xi\ \}$$

· If α is a limit ordinal,

$$\Sigma^\alpha = \bigcap_{\beta<\alpha} \Sigma^\beta.$$

Notice that $\Sigma^\alpha \subseteq \Sigma^\beta$. if $\alpha > \beta$. The *rank* of Σ, denoted $\operatorname{rank}(\Sigma)$, is the smallest ordinal α such that $\Sigma^{\alpha+1} = \emptyset$. If such an ordinal does not exist, we say that Σ has *unbounded rank* and write $\operatorname{rank}(\Sigma) = \infty$.

15.1**. Proposition.** *Suppose that* $\operatorname{rank}(\Sigma) = \infty$. *Then there exists a sequence* (ξ_n) *in* Σ *such that* $\xi_0 < \xi_1 < \dots$

Proof. Fix an ordinal α such that $\Sigma^\alpha = \Sigma^\beta$ for every $\beta > \alpha$. Take $\xi_0 \in \Sigma^\alpha$. Then $\xi \in \Sigma^{\alpha+1}$, so there exists $\xi_1 \in \Sigma^\alpha$ with $\xi_1 > \xi_0$. Now, $\xi_1 \in \Sigma^{\alpha+1}$, so there exists $\xi_2 \in \Sigma^\alpha$ with $\xi_2 > \xi_1$. Continuing in this fashion, we find (ξ_n) as desired. ⊣

Let $X^{<\omega}$ denote the set of finite sequences of X. If $\xi, \eta \in X^{<\omega}$, we write $\xi < \eta$ if η extends ξ.

15.2**. Proposition.** *Suppose that* X *is stable. Then there exists* $p \in [1, \infty]$ *such that for every* $\epsilon > 0$, *the set*

$$\left\{ \xi \in X^{<\omega} \;\middle|\; \xi \textit{ is } (1+\epsilon)\textit{-equivalent to the standard unit basis of } \ell_p(n), \textit{ for some } n < \omega \right\}.$$

has unbounded rank.

Before proving the proposition, let us invoke it to prove the following famous result.

15.3**. Theorem** (Krivine-Maurey, 1980)**.** *For every stable Banach space* X *there exists a number* $p \in [1, \infty)$ *such that for every* $\epsilon > 0$ *there exists a sequence in* X *which is* $(1 + \epsilon)$*-equivalent to the standard unit basis of* ℓ_p.

Proof. By Propositions 15.1 and 15.2, there exists $p \in [1, \infty]$ such that for every $\epsilon > 0$ there exists a sequence in X which is $(1 + \epsilon)$-equivalent to the standard unit basis of ℓ_p. But the stability of X rules out the case $p = \infty$ (see Remark 13.4), so the theorem follows. ⊣

Proof of Proposition 15.2. Use Proposition 14.3 to fix a symmetric type t_0 over X of norm 1 and such that $\mathfrak{p}[t] = \mathfrak{p}[t_0]$ for every type $t \in \overline{[t_0]}$ of norm 1. Fix $p \in \mathfrak{p}[t]$ and let

$$\Sigma[p, \epsilon] = \left\{ \xi \in X^{<\omega} \;\middle|\; \xi \text{ is } (1+\epsilon)\text{-equivalent to the standard unit basis of } \ell_p(n), \text{ for some } n < \omega \right\}.$$

For the sake of argument, assume $p < \infty$. (If $p = \infty$, the notational changes required below are obvious.)

We construct for every ordinal α a type t_α over X such that

(1) $\|t_\alpha\| = 1$;
(2) t_α is symmetric;
(3) $t_\alpha \in \overline{[t_\beta]}$ for every $\beta < \alpha$;

(4) For every $\epsilon > 0$, every finite dimensional subspace E of X, and every element c with $\mathrm{tp}(c/X) \in [t_\alpha]$, the set

$$\Sigma[\epsilon, E, c] =$$
$$\Big\{ \ (x_0, \dots, x_n) \in X^{<\omega} \ \Big| \ \mathrm{tp}\,\big(\sum_{i=0}^{n} \lambda_i x_i \ / \ E \ \big) \ \overset{1+\epsilon}{\sim} \ \big(\sum_{i=0}^{n} |\lambda_i|^p \big)^{1/p} \, \mathrm{tp}(c/E)$$
$$\text{whenever } \lambda_0, \dots, \lambda_n \text{ are scalars} \Big\}$$

has rank $\geq \alpha$.

Notice that if $(x_0, \dots, x_n) \in \Sigma[\epsilon, E, c]$ and $c \neq 0$, then

$$\Big(\frac{x_0}{\|c\|}, \dots, \frac{x_n}{\|c\|} \Big) \in \Sigma\big[\epsilon, E, \frac{c}{\|c\|}\big].$$

Hence, condition (4) ensures that $\mathrm{rank}(\Sigma[p, \epsilon]) = \infty$. The other conditions are set to allow the inductive construction to go through.

Note that (3) implies that $p \in \mathfrak{p}[t_\alpha]$ for every ordinal α.

The type t_0 defined above satisfies (1)–(3). Condition (4) is immediate from the symmetry of t and the fact that every approximation of a type over X (in the sense of Section 1) is realized in any finite dimensional subspace of X.

Suppose that t_α has been defined, let (u_l) be a sequence of types of norm 1 in $[t_\alpha]$ which witnesses the fact that $p \in \mathfrak{p}[t_\alpha]$, and define $t_{\alpha+1} = \lim u_l$. Conditions (1)–(3) are clearly satisfied. We prove (4).

Fix $\epsilon > 0$, a finite dimensional subspace E of X. Take real numbers δ_1, δ_2 such that $0 < \delta_1 < \delta_2 < \epsilon$ and $(1 + \delta_2)^2 < 1 + \epsilon$.

For scalars $r_0, \dots, r_n$, we have

$$(\dagger) \qquad \big(\sum_{i=0}^{n} |r_i|^p \big)^{1/p} t_\alpha = \lim_l \, (r_0 u_l * \cdots * r_n u_l).$$

Fix an element c such that $\mathrm{tp}(c/X) \in [t_{\alpha+1}]$. Each u_l is in $[t_\alpha]$, so using (†) and the fact that the convolution is commutative and separately continuous, for every $x \in X$ we find types $w_0, \dots, w_n \in [t_\alpha]$ such that

$$\big(\sum_{i=0}^{n} |r_i|^p \big)^{1/p} \mathrm{tp}(c/X)(x) \ \overset{1+\delta_1}{\sim} \ r_0 w_0 * \cdots * r_n w_n(x)$$

for all scalars $r_0, \dots, r_n$. Let d be a realization of w_0. Since E is finite dimensional, there exist $y_1, \dots, y_n \in X$ such that

$$(\ddagger) \qquad \big(\sum_{i=0}^{n} |r_i|^p \big)^{1/p} \mathrm{tp}(c/E) \ \overset{1+\delta_2}{\sim} \ \mathrm{tp}\,\big(r_0 d + \sum_{i=1}^{n} r_i y_i \ / \ E \big)$$

for all scalars $r_0, \dots, r_n$. Let

$$F = \overline{\mathrm{span}}\big\{ \, E \cup \{y_1, \dots, y_n, \} \, \big\}.$$

We now prove that

$$(x_0, \dots, x_n) \in \Sigma[\delta_2, F, d] \quad \text{implies} \quad (x_0, \dots, x_n, y_1, \dots, y_n) \in \Sigma[\epsilon, E, c].$$

Since $\text{tp}(d/X) = w_0 \in \text{span}(t_\alpha, *)$, this will conclude the proof of (4). Fix scalars $\lambda_0, \dots, \lambda_n, \mu_1, \dots, \mu_n$, and suppose $(x_0, \dots, x_n) \in \Sigma[\delta_2, F, d]$. Then we have

$$\text{tp}\big(\sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i y_i \ / \ E\big) \overset{1+\delta_2}{\sim} \text{tp}\big(\big(\sum_{i=0}^{n} |\lambda_i|^p\big)^{1/p} d + \sum_{i=1}^{n} \mu_i y_i \ / \ E\big)$$

Hence, by (‡),

$$\text{tp}\big(\sum_{i=0}^{n} \lambda_i x_i + \sum_{i=1}^{n} \mu_i y_i \ / \ E\big) \overset{(1+\delta_2)^2}{\sim} \big(\sum_{i=0}^{n} |\lambda_i|^p + \sum_{i=1}^{n} |\mu_i|^p\big)^{1/p} \text{tp}(c/E)$$

Since $(1+\delta_2)^2 < 1+\epsilon$, it follows that $(x_0, \dots, x_n, y_1, \dots, y_n) \in \Sigma[\epsilon, E, c]$.

If α is a limit ordinal, we take an ultrafilter $\mathcal{U}$ on α and define $t_\alpha = \lim_{\beta<\alpha, \mathcal{U}} t_\beta$. ⊣

16. Historical Remarks

Section 1: The construction of Banach space ultrapower was explicitly introduced by D. Dacunha-Castelle and J. L. Krivine in [12] (although ultrapowers had been used by Krivine and others in earlier publications; see [11]) based on the ultraproduct construction from model theory. The Banach space ultrapower construction is a particular case of the *nonstandard hull* construction introduced by W. A. J. Luxemburg in [52].

For a survey on applications of nonstandard hulls to Banach space theory, see [38]. The classical reference for Banach space ultrapowers is [29]. A somewhat more recent survey is [73].

The logic of positive bounded formulas and approximate satisfaction was introduced by C. W. Henson in [35]. The precursor was [34]. (See also [30, 31, 32, 33, 37].) In the general framework of Banach space model theory, one considers structures of the form

$$(X, \ R_i, \ f_j, \ c_k \ \mid \ i \in I, \ j \in J, \ k \in K \ ),$$

where the c_k's are constants, the f_j's are functions from X^n into X, (for some n depending on j), and the R_i's are *real-valued relations*, *i.e.*, functions from X^n (for some n) into the extended real numbers. The functions and real-valued relations are required to be uniformly continuous on every bounded subset of X, and the language is required to come equipped with norm bounds for the constants and moduli of uniform continuity for the functions and real-valued relations on each bounded subset of X. One does not generally deal with ultrapowers, but rather with general models.

The notion of $(1+\epsilon)$-approximation and Theorem 1.10 are due to S. Heinrich and C. W. Henson [30].

In this section we have discussed only the most basic aspects of Banach space model theory. For more advanced aspects of the theory, *e.g.*, forking and stability, see [43, 44, 41, 42].

Related, but less general approaches to Banach spaces as models were proposed by J.-L. Krivine [47, 48] and J. Stern [74].

Section 2: The notions of splitting and semidefinability in model theory are due to S. Shelah, and the results in this section are straightforward adaptations of results in [70].

Sections 3 and 4: In model theory, a standard method of producing indiscernible sequences inside a stable structure by taking Morley sequences of a type, *i.e.*, sequences that result from realizations of successive nonforking (*i.e.*, semidefinable) extensions of the type. The assumption of stability ensures that these sequences are uniquely determined by the type that one starts with. When the model is unstable, the uniqueness is generally lost, but one can preserve it by working extensions of the type over sufficiently rich sets of parameters. (This was exploited by Shelah [70] and further by R. Grossberg [26]).

In Banach space theory, notions of "type" and "stability", which under a suitable translation correspond to the quantifier-free versions of type and stability in Banach space model theory, were introduced by J.-L. Krivine and B. Maurey in [50]; see the remarks on Section 5 below. Later, in [53], Maurey generalized some of the ideas from 5 to unstable contexts, and introduced a notion of "strong type" which can be seen as a quantifier-free version of the concept of strong type defined in [50]. The goal of [53] is to give a characterization of the Banach spaces that contain ℓ_1 (a similar characterization is given for c_0). In the introduction, the author writes: "This approach perhaps makes the proof unnecessarily long, but we feel that the notion of strong type can be useful".

Our proof of existence of symmetric Maurey strong types via types using the Borsuk-Ulam theorem is an elaboration of an idea in [66]. See also [68] and [53].

Section 4: The term "fundamental sequence" is borrowed from the theory of spreading models. See the remarks on Section 7 below.

Section 5: The definition of "type" in analysis was introduced in [50]. The definition in [50] is as follows. A separable Banach space X is fixed. If $a \in X$, the function $\tau_a \colon X \to \mathbb{R}$ is defined by $\tau_a(x) = \|a + x\|$. The *space of types* is the closure of the set $\{ \tau_a \mid a \in X \}$ in the product space $\mathbb{R}^X$. Proposition 5.2 shows that the space of types in this sense is exactly the space of quantifier-free types over X.

For further applications of the concept of type to Banach space geometry, see for example [8, 15, 27, 28, 53, 55, 61, 62, 63, 67, 68].

The definition of "approximating sequence" is also given in [18]; it appears there, however, without the clause "over X", since there, the space X is regarded as fixed throughout.

Section 6: For a survey on applications of Ramsey's Theorem to Banach space geometry, see [54].

Powerful strengthenings of Ramsey's Theorem due to W. T. Gowers and B. Maurey have led to the construction of Hereditarily Indecomposable spaces and to a chain of some of the most spectacular breakthroughs in the history of Banach space theory. For a nontechnical exposition, see [21] and [58]. The paper [22] contains a more recent although more technical survey. Further remarks on these important developments are at the end of this paper.

Section 7: Spreading models were introduced in analysis by A. Brunel and L. Sucheston [4, 5] in the study of summability of sequences in Banach spaces. The authors proved in [5] that whenever (x_n) is a bounded sequence in a Banach space X there exists a subsequence (x'_n) of (x_n) such that the limit

$$\lim_{n'_0<\cdots<n'_k} \|r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x\|$$

exists for every $r_0, \ldots, r_k \in \mathbb{R}$. The sequence (x'_n) is called a *good subsequence* of (x_n). We outline the argument of Brunel and Sucheston. A good subsequence (x'_n) induces a seminorm on $\mathbb{R}^\omega$ (or $\mathbb{C}^\omega$ if the space X is complex) as follows. If (e_n) is the standard basis of unit vectors in $\mathbb{R}^\omega$,

$$\|\sum_i r_i e_i\| = \lim_{n'_0<\cdots<n'_k} \|r_0 x'_{n_0} + \cdots + r_k x'_{n_k} + x\|.$$

This seminorm is a norm if (and only if) the sequence (x'_n) is nonconvergent. The resulting Banach space is called the *spreading model defined by the sequence* (x_n). The clause "over X" is not used by analysts, since the space X is normally regarded as fixed. The sequence (e_n) above is called the *fundamental sequence* of the model. It should be remarked that, despite this terminology, neither the good sequence (x'_n) nor the sequence (e_n) are uniquely determined by (x_n).

Analysts use the term 1-*subsymmetric* to express the fact a sequence in a Banach space is indiscernible (with respect to quantifier-free formulas).

J.-L. Krivine constructed spreading models in [49] using iterated Banach space ultrapowers. Both constructions are presented in detail in [2].

Section 8: ℓ_p- and c_0-types were introduced in [50] in the context of (quantifier-free) stable Banach spaces. In the general context (*i.e.*, without the assumption of stability), these notions can be defined only for Maurey strong types, since it is for those that a one has a well defined convolution which extends vector addition.

Section 9: The simplification of the proof of Krivine's Theorem through the use of eigenvectors of operators (Proposition 9.2) is due to H. Lemberg [51]. See the comments on Sections 11 and 12 for further remarks on Lemberg's proof.

Section 11: Our proof of Theorem 11.1 is based on H. Lemberg's proof of Krivine's Theorem [51]. We have tried to highlight the fact that, from a model theoretical perspective, the main idea is quite natural.

For a long time, it was an open problem whether every Banach space has a spreading model containing ℓ_p $(1 \le p < \infty)$ or c_0. The question was answered negatively by E. Odell and Th. Schlumprecht in [59]. In the same paper, the authors also provided an example of a space with an unconditional basis for which ℓ_p and c_0 are block-finitely represented in all block bases.

Section 12: The original statement of Krivine's Theorem in [49] was that given any bounded sequence (x_n) in a Banach space, either there exists p with $1 \le p < \infty$ such that ℓ_p is block finitely represented in (x_n), or there exists a permutation of (x_n) such that c_0 is block finitely represented in (x_n). In [65], H. P. Rosenthal expounded Krivine's Theorem and showed that the permutation of (x_n) in the c_0 case was unnecessary. In [51], H. Lemberg extracted the essential aspects of Rosenthal's proof, and simplified the argument further by using Proposition 9.2.

Section 14: Proposition 14.3 is from [6], and it plays a role analogous to that played by *minimal cones* in [50].

Section 15: The question of what Banach spaces contain ℓ_p or c_0 almost isometrically has played a central in the history of Banach space geometry. The first example of a Banach space not containing ℓ_p or c_0 (not even isomorphically) was constructed by B. S. Tsirel'son [75]. This phenomenon was even more dramatic for the dual of the original Tsirel'son space [16], which later became also known as the Tsirel'son space and has been used as cornerstone for further variations of the original. *Tsirel'son spaces* became an object of rather intense study. (See [7].)

In 1981, using probabilistic methods, D. Aldous proved [1] that every subspace of L_1 contains ℓ_p or some ℓ_p $(1 \leq p < \infty)$ almost isometrically. Almost immediately, J.-L. Krivine and B. Maurey generalized the methods of Aldous to a wider class of spaces: the class of stable Banach spaces. The role played by types in [50] (regarded as real-valued functions, see the notes on Section 5 above) is analogous to that played by random measures in Aldous' proof.

A wealth of examples of stable Banach spaces is exhibited in [50]. Furthermore, the authors provide methods to construct new stable Banach spaces from old ones; specifically, it is proved that if X is stable, then the space $L_p(X)$ is stable, for $1 \leq p < \infty$. Further examples are given in [18] and [60].

The general theory of model theoretical stability for Banach space structures (*e.g.*, forking, stability spectrum, etc.) was developed in [40]. See [43, 44, 41].

Our proof of Theorem 15.3 is based on a proof by S. Q. Bu [6]. In [6], Bu invokes a principle from descriptive set theory that C. Dellacherie in [13] labelled *the Kunen-Martin Theorem*. Bu proves Theorem 15.3 by showing that there are types of arbitrarily high countable rank. Our argument shows that one need not invoke the Kunen-Martin Theorem if one considers values on all ordinals, rather than the countable ones.

For an important application of ordinal ranks in Banach space theory, we refer the reader to [3].

F. Chaatit [9] showed that a Banach space is stable if and only if it can be embedded in the group of isometries of a reflexive Banach space.

It was noticed by Krivine and Maurey that if X is a stable Banach space, then the space of types over X is *strongly separable*, *i.e.*, separable with respect to the topology of uniform convergence on bounded subsets of X (recall that for Banach space theorists stable spaces are by definition separable, and types are real-valued functions; see the notes on Section 5 above). E. Odell proved (see [55] or [62]) that strong separability of the space of types does not imply stability by showing that the space of types over the Tsirel'son space of [16] is strongly separable. Later, in [28], R. Haydon and B. Maurey proved that every space with a strongly separable space of types contains either a reflexive subspace or a copy of ℓ_1. In [45], the author identified topological conditions on the space of types of a Banach space that characterize stability of the space.

We conclude by remarking that the decade of the 1990's was a time of historical developments in Banach space theory. Many of the most famous problems of the field (some of which had remained open since Banach's time) were finally solved. The key lay in a deeper understanding of Tsirel'son's space.

Based on a construction of Th. Schlumprecht [69], Gowers and Maurey [25] constructed a *Hereditarily Indecomposable space*, *i.e.*, a Banach space such that no subspace X of it is isomorphic to a sum of two infinite dimensional subspaces of X. The authors proved that a Hereditarily Indecomposable space does not contain an unconditional basic sequence (*i.e.* no sequence (x_n) satisfying $\|\sum \theta_n r_n x_n\| \leq K\|\sum r_n x_n\|$ for some $K > 0$, and all scalars r_n for which $\sum r_n x_n$ converges, and all θ_n with $|\theta_n| = 1$), thus solving the Unconditional Base Problem. The authors also proved that a Hereditarily Indecomposable space cannot be isomorphic to any of its subspaces, and therefore it cannot be isomorphic (let alone isometric) to any of its hyperplanes. This solves Banach's Hyperplane Problem. (Gowers had just presented a solution to the Hyperplane Problem in [19].)

Later, Gowers [20] refined the techniques of [25] to exhibit a space that contains no isomorphic copy of c_0, ℓ_1, or an infinite dimensional reflexive space, answering a long standing question.

More recently [23], Gowers solved negatively the Schroeder-Bernstein Problem by exhibiting two nonisomorphic Banach spaces that are isomorphic to complemented subspaces of each other. The construction is based on the space with no unconditional basic sequence provided in [25].

In [22], using topological games and sophisticated forms of Ramsey's Theorem, Gowers provided the final positive solution to Mazur's Homogeneous Space Problem. A space is homogeneous if it is isomorphic to all of its infinite dimensional subspaces. Gowers shows in [22] that any Banach space either has a subspace with an unconditional basis, or contains a Hereditarily Indecomposable subspace. Hence a homogeneous space must have an unconditional basis, and by a result of R. Komorowski and N. Tomczak-Jaegermann [46], it must be isomorphic to ℓ_2.

A Banach space $(X, \|\ \|)$ is said to be *distortable* if there exist an equivalent norm $|\ |$ on X and a $\epsilon > 0$ such for every infinite dimensional Y of X we have

$$\sup\{\ |y|/|x|\ \mid\ x, y \in Y,\ \|x\| = \|y\| = 1\ \} > 1 + \epsilon.$$

The Distortion Problem is whether every Hilbert space is distortable. In [57], E. Odell and Th. Schlumprecht solved affirmatively the Distortion Problem. (The solution had been announced earlier in [56]. See also [58].) Furthermore, the authors proved that any space not containing an isomorphic copy of ℓ_1 or c_0 contains a distortable subspace.

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