A FINITE VOLUME IMPLICIT EULER SCHEME FOR THE LINEARIZED SHALLOW WATER EQUATIONS: STABILITY AND CONVERGENCE

K. Adamy □ Laboratoire d’Analyse Numérique, Université Paris-Sud, Orsay, France
D. Pham □ The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, Indiana, USA

The linearized shallow water equations are discretized in space by a finite volume method and in time by an implicit Euler scheme. Stability and convergence of the scheme are proved.

Keywords Convergence; Finite volumes; Implicit Euler scheme; Linearized shallow water equations; Stability.

AMS Subject Classification 65M12; 86A05.

1. INTRODUCTION

In this article, the linearized shallow water equations (SW) are discretized in space by a finite volume method (FV) and in time by the implicit Euler scheme, which is found in the form: To find recursively the \( u^n_h \) with \( u_0^h \) given and

\[
\frac{u^n_h - u^{n-1}_h}{\Delta t} + A_h u^n_h = f^n_h, \tag{1.1}
\]

where \( A_h \) is the FV discrete operator, which is proved to be positive on the FV space \( V_h \) with the use of the centered FV approximation, see Lemma 3.3. We then prove that the scheme (1.1) is stable and convergent. The proof of stability benefits from the positivity of the FV operator \( A_h \). For simplicity,
we will briefly present the full details of the finite volume discretization as well as the stability proof for an implicit Euler scheme for the 1D transport equation. The FV discretization technique and the stability proof in this simple case will then be used for the case of the SW equations, in dimension two.

Our work is as follows: we discretize the 1D transport equation by finite volumes with time discretization performed by an implicit Euler scheme and prove the stability of this scheme in Section 2. In Section 3, we discretize the SW equations by finite volumes. Then we prove that the discretized FV operator $A_h$ for these SW equations is positive, and hence the same stability result as in Section 2 can be applied. Finally, we give the proof of convergence for the finite volume discretization in space and implicit Euler scheme discretization in time for the SW equations in Section 4.

2. IMPLICIT EULER FV SCHEMES FOR $u_t + \alpha u_x = f$ AND STABILITY

In this section, we discretize the 1D transport equation by finite volumes in space and implicit Euler scheme in time. We thus present in this simpler case the technique of discretization and the details of the proof of stability that will be used under higher dimension for the linearized shallow water equations.

The equation under consideration is as follows:

$$\begin{cases}
u_t(x, t) + \alpha u_x(x, t) = f(x, t), & x \in \mathcal{M}, \ t > 0, \\
u(0, t) = 0, & t > 0, \\
u(x, 0) = u_0(x), & x \in \mathcal{M},
\end{cases} \quad (2.1)$$

where $\mathcal{M} = (0, 1)$, $t \in (0, T)$, and $\alpha > 0$.

We set $H = L^2(\mathcal{M})$, $D(A) = \{v \in H^1(\mathcal{M}), v(0) = 0\}$, and $D(A^*) = \{v \in H^1(\mathcal{M}), v(1) = 0\}$. Thanks to the boundary condition $u(0) = 0$, the operator $Au = u_x$ is positive on $D(A)$. Hence for $f, f' \in L^1(0, T, H)$, $u_0 \in D(A)$, by an application of the Hille–Yosida theorem (see [15]), equation (2.1) possesses a unique solution $u$ such that

$$u \in C(0, T; H) \cap L^\infty(0, T; D(A)), \quad \frac{du}{dt} \in L^\infty(0, T; H). \quad (2.2)$$

We note that the boundary condition type set that leads to this well-posedness was already proposed in the papers [9, 10, 13] and was also used to prove the stability results for various numerical schemes in [7].
2.1. FV Discretization

First, we introduce the uniform FV mesh for \( \mathcal{M} \). We choose the meshes \( h_1, \ldots, h_N \) and divide \( \mathcal{M} \) into \( N \) subintervals \( K_i = (x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}}), \ i = 1, \ldots, N \) where \( x_\frac{1}{2} = 0, x_1 = h_1, x_2 = h_1 + h_2, \ldots, x_{N+\frac{1}{2}} = \sum_{i=1}^{N} h_i \). The \( K_i \)'s are called the control volumes, and for each \( K_i \) we denote by \( x_i \) the center of it. For more details about the FV mesh, see [3].

We then introduce the following FV space approximations of \( H, D(A) \), and \( D(A^*) \):

\[
V_h = \text{Space of step functions constant on } K_i, \quad i = 1, \ldots, N, \quad u_{i|K_i} = u_i. \tag{2.3}
\]

\[
V_{0h} = \text{Space of step functions constant on } K_i, \quad i = 0, \ldots, N + 1, \quad u_{i|K_i} = u_i, \quad u_0 = -u_1, \quad u_{N+1} = u_N. \tag{2.4}
\]

\[
V_{1h} = \text{Space of step functions constant on } K_i, \quad i = 0, \ldots, N + 1, \quad u_{i|K_i} = u_i, \quad u_{M+1} = -u_M, \quad u_0 = u_1. \tag{2.5}
\]

Note that \( V_h, V_{0h}, V_{1h} \) are linearly isomorphic, but we make a different use of these spaces. The support of the functions in \( V_h \) is \( \mathcal{M} \) while the support of the functions in \( V_{0h} \) and \( V_{1h} \) strictly contains \( \mathcal{M} \). In the following, if the supports of the functions are larger than \( \mathcal{M} \), then their scalar products or norms are the scalar products or norms in \( H = L^2(\mathcal{M}) \) of their restrictions to \( \mathcal{M} \).

**Scalar product and norm on** \( V_h \):

\[
(\cdot, \cdot)_h = (\cdot, \cdot)_H, \quad |\cdot|_h = |\cdot|_H. \tag{2.6}
\]

**Scalar products and norms on** \( V_{0h}, V_{1h} \): Let \( u_h, \tilde{u}_h \in V_{0h} \) (or \( V_{1h} \)), we first define the discrete FV derivative of \( u_h \) (similarly for \( \tilde{u}_h \)) by setting

\[
\nabla_h u_h = \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \text{ on } K_{i+\frac{1}{2}}, \quad i = 0, \ldots, N. \tag{2.7}
\]

Here

\[
K_{i+\frac{1}{2}} := (x_i, x_{i+1}),
\]

\[
h_{i+\frac{1}{2}} := x_{i+1} - x_i, \quad i = 0, \ldots, N, \quad x_0 = -x_1, \quad x_{N+1} = x_N + h_N.
\]

The scalar product of \( u_h, \tilde{u}_h \) and norm of \( u_h \) in \( V_{0h} \) are then defined as follows:

\[
((u_h, \tilde{u}_h))_h = (\nabla_h u_h, \nabla_h \tilde{u}_h)_H, \quad \|u_h\|_h = |\nabla u_h|_H. \tag{2.8}
\]
To derive the discretization of (2.1) by FV, we integrate (2.1) over $K_i$ and divide the equation by $h_i$, for $1 \leq i \leq N$; we obtain

$$\frac{du_i}{dt} + \frac{\alpha}{h_i} (u(x_{i+\frac{1}{2}}) - u(x_{i-\frac{1}{2}})) = f_i,$$

where $u_i = h_i^{-1} \int_{K_i} u(x) \, dx$.

To approximate the fluxes $(u(x_{i+\frac{1}{2}}) - u(x_{i-\frac{1}{2}}))/h_i$, we write

$$u(x_{i+\frac{1}{2}}) \approx \gamma_{i+\frac{1}{2}} u_{i+1} + (1 - \gamma_{i+\frac{1}{2}}) u_i, \quad \text{for } \gamma_{i+\frac{1}{2}} \in [0, 1].$$

We note here that because $\alpha > 0$, we obtain an upstream approximation if $\gamma_{i+\frac{1}{2}} = 0$ and a downstream approximation if $\gamma_{i+\frac{1}{2}} = 1$. We write

$$u(x_{i+\frac{1}{2}}) - u(x_{i-\frac{1}{2}}) \approx \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) + (1 - \gamma_{i+\frac{1}{2}}) (u_i - u_{i-1})$$

$$= \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) + u_i - (\gamma_{i-\frac{1}{2}} (u_i - u_{i-1}) + u_{i-1})$$

$$= \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) + (1 - \gamma_{i-\frac{1}{2}}) (u_i - u_{i-1}).$$

Hence, the proposed FV scheme becomes

$$\frac{du_i}{dt} + \frac{\alpha}{h_i} \left( \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) + (1 - \gamma_{i-\frac{1}{2}}) (u_i - u_{i-1}) \right) = f_i, \quad (2.9)$$

for $i = 1, \ldots, N$ with the notation convention that $u_0 = -u_1$, $u_{N+1} = u_N$.

Let $A_h : V_{0h} \to V_h$ be defined by

$$A_h u_h = \frac{\gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) + (1 - \gamma_{i-\frac{1}{2}}) (u_i - u_{i-1})}{h_i} \quad \text{on } K_i, \quad 1 \leq i \leq N. \quad (2.10)$$

Then (2.9) is equivalent to

$$\frac{du_h}{dt} + \alpha A_h u_h = f_h, \quad (2.11)$$

The following lemma shows that the operator $A_h$ is positive, hence is maximal monotone on $V_{0h}$.

**Lemma 2.1.** For $u_h \in V_{0h}$, we have $(A_h u_h, u_h)_h \geq 0$ provided

$$\gamma_{i+\frac{1}{2}} \leq \frac{3}{4}, \quad \gamma_{i+\frac{1}{2}} \leq \frac{1}{2}, \quad 1 \leq i \leq N - 1. \quad (2.12)$$
**Proof.** Let $u_h \in V_{0h}$; we have

$$
(A_h u_h, u_h)_h = \sum_{i=1}^{N} \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) + \left(1 - \gamma_{i-\frac{1}{2}}\right) (u_i - u_{i-1}) u_i h_i
$$

$$
= \sum_{i=1}^{N} \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i) u_i + \sum_{i=1}^{N} \left(1 - \gamma_{i-\frac{1}{2}}\right) (u_i - u_{i-1}) u_i.
$$

Using

$$
2(a - b) a = a^2 - b^2 + (a - b)^2,
$$

$$
2(a - b) b = a^2 - b^2 - (a - b)^2,
$$

we find

$$
2(A_h u_h, u_h)_h = \sum_{i=1}^{N} \gamma_{i+\frac{1}{2}} ((u_{i+1})^2 - (u_i)^2 - (u_{i+1} - u_i)^2)
$$

$$
+ \sum_{i=1}^{N} \left(1 - \gamma_{i-\frac{1}{2}}\right) ((u_i)^2 - (u_{i-1})^2 + (u_i - u_{i-1})^2)
$$

$$
= \sum_{i=1}^{N} \gamma_{i+\frac{1}{2}} ((u_{i+1})^2 - (u_i)^2 - (u_{i+1} - u_i)^2)
$$

$$
- \sum_{i=1}^{N} \gamma_{i-\frac{1}{2}} ((u_i)^2 - (u_{i-1})^2 + (u_i - u_{i-1})^2)
$$

$$
+ \sum_{i=1}^{N} ((u_i)^2 - (u_{i-1})^2 + (u_i - u_{i-1})^2).
$$

Because $u_{N+1} = u_N$, we have

$$
2(A_h u_h, u_h)_h = \sum_{i=1}^{N-1} \gamma_{i+\frac{1}{2}} ((u_{i+1})^2 - (u_i)^2 - (u_{i+1} - u_i)^2)
$$

$$
- \sum_{i=0}^{N-1} \gamma_{i+\frac{1}{2}} ((u_{i+1})^2 - (u_i)^2 + (u_{i+1} - u_i)^2)
$$

$$
+ \sum_{i=0}^{N-1} ((u_{i+1})^2 - (u_i)^2 + (u_{i+1} - u_i)^2)
$$

$$
= -\gamma_{\frac{1}{2}} ((u_1)^2 - (u_0)^2 + (u_1 - u_0)^2) - 2 \sum_{i=1}^{N-1} \gamma_{i+\frac{1}{2}} (u_{i+1} - u_i)^2
$$

$$
+ \sum_{i=0}^{N-1} ((u_{i+1})^2 - (u_i)^2 + (u_{i+1} - u_i)^2)$$

Because $u_{N+1} = u_N$, we have
\[ (1 - \gamma^2) ((u_i)^2 - (u_0)^2 + (u_1 - u_0)^2) \]
\[ + \sum_{i=1}^{N-1} \left( 1 - 2\gamma_{i+\frac{1}{2}} \right) (u_{i+1} - u_i)^2 + \sum_{i=1}^{N-1} (u_{i+1})^2 - (u_i)^2. \]

Due to the boundary condition \( u_0 = -u_1 \), we see that
\[ 2(A_h u_h, u_h)_h = (3 - 4\gamma) (u_1)^2 + \sum_{i=1}^{N-1} \left( 1 - 2\gamma_{i+\frac{1}{2}} \right) (u_{i+1} - u_i)^2 + (u_N)^2. \]

By the hypotheses (2.12), we conclude that \((A_h u_h, u_h)_h \geq 0\). \( \square \)

We consider the implicit Euler for (2.11): to find recursively \( u^n_h \) such that
\[
\frac{u^n_h - u^{n-1}_h}{\Delta t} + \alpha A_h u^n_h = f^n_h, \tag{2.13}
\]
where \( u^0 = r_h u_0 = h^{-1} \int_{K_i} u_0(x) \, dx \) on \( K_i \) and \( n = 1, \ldots, N_i := T/\Delta t \). The operator \( r_h \) is called the restriction operator; for more information about this operator, see [12].

Let us denote by \( I_h \) the identity map on \( V_h \) which can be restricted to \( V_{0h} \). The functional form of (2.13) is found to be:
\[
\frac{I_h u^n_h - I_h u^{n-1}_h}{\Delta t} + \alpha A_h u^n_h = f^n_h, \quad n = 1, \ldots, N_i. \tag{2.14}
\]

**Lemma 2.2.** The solutions of (2.14) satisfy
\[
A_h u^n_h = (I_h + \Delta t \alpha A_h)^{-n} A_h u_{0h} - \sum_{m=2}^{n} (I_h + \Delta t \alpha A_h)^{-(n-m)} (f^m_h - f^{m-1}_h)
\]
\[ - (I_h + \Delta t \alpha A_h)^{-(n-1)} f^1_h. \tag{2.15} \]

**Proof.** Write (2.14) as
\[
(I_h + \Delta t \alpha A_h) u^n_h = u^{n-1}_h + \Delta t f^n_h;
\]
we have
\[
u^n_h = (I_h + \Delta t \alpha A_h)^{-1} (u^{n-1}_h + \Delta t f^n_h)
\]
\[ = (I_h + \Delta t \alpha A_h)^{-1} \{(I_h + \Delta t \alpha A_h)^{-1} u^{n-2}_h + \Delta t (I_h + \Delta t \alpha A_h)^{-1} f^{n-1}_h + \Delta t f^n_h} \]
\[ \vdots \]
\[ = (I_h + \Delta t \alpha A_h)^{-n} u^0_h + \Delta t \sum_{m=0}^{n} (I_h + \Delta t \alpha A_h)^{-m} f^{n-m}_h. \]
Lemma 2.3. Under the assumption (2.12), and if $\Delta t \leq x^*$ where $x^* (\approx 0.7968)$ is the positive root of $f(x) = e^{-2x} + x - 1$, we have for all $1 \leq n \leq N_t$ that

\[
|u^n_h|_h^2 \leq e^{2T} \left( |r_h u_0|_h^2 + \sup_{m=1,\ldots,N_t} |f^m_h|_h^2 \right),
\]

(2.16)

\[
|A_h u_h|_h \leq |A_h(r_h u_0)|_h + \sum_{m=2}^{n} |f^m_h - f^{m-1}_h|_h + |f^1_h|_h.
\]

(2.17)

Proof. For $n \in \{1, \ldots, N_t\}$, we take the scalar product of (2.14) with $2\Delta t u^n_h$ and have

\[
2(u^n_h - u^{n-1}_h, u^n_h)_h + 2\Delta t (A_h u^n_h, u^n_h)_h = 2\Delta t (f^n_h, u^n_h)_h.
\]

(2.18)

Using $2(u - v, u)_h = |u|^2_h - |v|^2_h + |u - v|^2_h$, we find

\[
|u^n_h|_h^2 - |u^{n-1}_h|_h^2 + |u^n_h - u^{n-1}_h|_h^2 + 2\Delta t (A_h u^n_h, u^n_h)_h = 2\Delta t (f^n_h, u^n_h)_h.
\]

We obtain by Lemma 2.1 that

\[
|u^n_h|_h^2 - |u^{n-1}_h|_h^2 + |u^n_h - u^{n-1}_h|_h^2 \leq 2\Delta t (f^n_h, u^n_h)_h.
\]

By the Cauchy–Schwarz inequality, we have

\[
2(f^n_h, u^n_h)_h \leq 2|f^n_h|_h u^n_h \leq |f^n_h|_h^2 + |u^n_h|_h^2.
\]
Hence

\[(1 - \Delta t) \left| \frac{u^n_h}{h} \right|^2 - \left| \frac{u^{n-1}_h}{h} \right|^2 + \left| u^n_h - u^{n-1}_h \right|_h^2 \leq \Delta t \left| f^n_h \right|_h^2.\]

Therefore

\[
\left| \frac{u^n_h}{h} \right|^2 \leq \frac{1}{1 - \Delta t} \left| \frac{u^{n-1}_h}{h} \right|^2 + \Delta t \left| f^n_h \right|_h^2
\]

\[
\leq \cdots
\]

\[
\leq \frac{1}{1 - \Delta t}^n \left| \frac{u^0_h}{h} \right|^2 + \Delta t \left| f^1_h \right|_h^2 + \cdots + \Delta t \left| f^n_h \right|_h^2
\]

\[
\leq \frac{1}{1 - \Delta t}^n \left| \frac{u^0_h}{h} \right|^2 + \sup_{n=1, \ldots, N_t} \left| f^n_h \right|_h^2 \frac{1}{1 - \Delta t} \left( \frac{1}{1 - \Delta t}^{n-1} + \cdots + 1 \right)
\]

By the inequality \( e^{-2x} \leq 1 - x \) for every \( x \in [0, x^*] \), we obtain for \( \Delta t \leq x^* \) that

\[
\left| \frac{u^n_h}{h} \right|^2 \leq e^{2n\Delta t} \left( \left| \frac{u^0_h}{h} \right|^2 + \sup_{n=1, \ldots, N_t} \left| f^n_h \right|_h^2 \right) \leq e^{2T} \left( \left| \frac{u^0_h}{h} \right|^2 + \sup_{n=1, \ldots, N_t} \left| f^n_h \right|_h^2 \right).
\]

Hence (2.16) holds.

Now Lemma 2.2 implies

\[
\left| A_h u^n_h \right|_h \leq \left| (I_h + \Delta t A_h)^{-n} A_h u^0_h \right|_h + \sum_{m=2}^{n} \left| (I_h + \Delta t A_h)^{-(n-m)} (f^m_h - f^{m-1}_h) \right|_h
\]

\[
+ \left| (I_h + \Delta t A_h)^{-(n-1)} f^1_h \right|_h.
\]

Due to Lemma 2.1, \( A_h \) is positive (maximal monotone) and hence is an contraction (see [2])

\[
\left| (I_h + \Delta t A_h)^{-m} \right|_{\mathcal{L}(V_h)} \leq 1, \quad \forall m \in \mathbb{N}.
\]

This implies that (2.17) holds. \( \square \)

3. THE SW EQUATIONS AND THEIR FV DISCRETIZATION

In this section, we introduce the linearized shallow water equations in a convenient way so that the positivity of the discretized FV problems are
easily seen. The 2D linearized SW equations on \( \mathcal{M} = [0, L_1] \times [0, L_2] \) reads (see, e.g., [5, 6]):

\[
\begin{align*}
\frac{\partial u}{\partial t} + \tilde{u}_0 \frac{\partial u}{\partial x} + \tilde{v}_0 \frac{\partial u}{\partial y} + \frac{\partial \Phi}{\partial x} - fu &= 0, \\
\frac{\partial v}{\partial t} + \tilde{u}_0 \frac{\partial v}{\partial x} + \tilde{v}_0 \frac{\partial v}{\partial y} + \frac{\partial \Phi}{\partial y} + fu &= 0, \\
\frac{\partial \Phi}{\partial t} + \tilde{u}_0 \frac{\partial \Phi}{\partial x} + \tilde{v}_0 \frac{\partial \Phi}{\partial y} + \Phi_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0.
\end{align*}
\]

Here \((u, v)\) is the velocity, and \(\Phi = gz\) (where \(g \approx 9.81 \text{ms}^{-2}\)) is the potential height. The Coriolis parameter \(f\), the advecting velocities \(\tilde{u}_0, \tilde{v}_0\), and the mean geopotential height \(\Phi_0\) are constants. For the subcritical flow under consideration (the physically more important case), the constants \(\tilde{u}_0, \tilde{v}_0, \Phi_0\) are such that

\[
\tilde{u}_0 > 0, \quad \tilde{v}_0 > 0, \quad \tilde{u}_0^2 + \tilde{v}_0^2 < \Phi_0.
\]

To have the system diagonalized in the matrix for the derivative in \(x\), we use the change of variables \(\xi = u + \left(1/\sqrt{\Phi_0}\right) \Phi, \zeta = \sqrt{2}v, \eta = u - \left(1/\sqrt{\Phi_0}\right) \Phi\) and write \(k = k(u)\), \(l = l(u)\), \(m = m(u)\) with

\[
\begin{align*}
k &= \tilde{u}_0 \xi + \Phi_1 \zeta, \\
l &= \tilde{u}_0 \zeta + \Phi_1 (\xi - \eta), \\
m &= \tilde{u}_0 \eta - \Phi_1 \zeta, \\
\Phi_1 &= \sqrt{\Phi_0/2}, \quad \tilde{u}_0 = \tilde{u}_0 + \sqrt{2} \Phi_1, \quad \Phi_0 = \tilde{u}_0 - \sqrt{2} \Phi_1 < 0, \quad \tilde{f} = f/\sqrt{2}.
\end{align*}
\]

The new system in \(u = (\xi, \zeta, \eta)\) becomes

\[
\frac{du}{dt} + Au = f,
\]

where the forcing term \(f = (f_1, f_2, f_3)\) has been added for mathematical generality although it does not exist in (3.1); and \(Au = (A_1u, A_2u, A_3u)\) with

\[
\begin{align*}
A_1u &= \tilde{u}_0 \frac{\partial \xi}{\partial x} + \frac{\partial k}{\partial y} - \tilde{f} \zeta, \\
A_2u &= \tilde{u}_0 \frac{\partial \zeta}{\partial x} + \frac{\partial l}{\partial y} + \tilde{f} (\xi + \eta), \\
A_3u &= \tilde{u}_0 \frac{\partial \eta}{\partial x} + \frac{\partial m}{\partial y} - \tilde{f} \zeta.
\end{align*}
\]
We consider the following initial condition for (3.1)–(3.4):
\[ u(0) = u_0, \] (3.6)
where \( u_0 = u_0(x, y) := (\xi_0(x, y), \tilde{\zeta}(x, y), \eta_0(x, y)) \).

The boundary conditions under consideration are
\[
\begin{cases}
    \xi(0, y, t) = \xi(0, y, t) = \eta(L_1, y, t) = 0, \\
    k(x, 0, t) = m(x, 0, t) = \zeta(x, L_2, t) = 0.
\end{cases}
\] (3.7)

We define the spaces
\[ H = (L^2(\mathcal{M}))^3, \]
\[ D(A) = \{u = (\xi, \zeta, \eta) \in H \text{ such that } Au \in H \text{ and } \]
\[ \xi(0, y) = \zeta(0, y) = \eta(L_1, y) = 0, k(x, 0) = m(x, 0) = \zeta(x, L_2) = 0 \}. \] (3.8)

It can be shown (see, e.g., [4] and also [1]) that if \( u \) and \( Au \) belong to \( H = L^2(\mathcal{M})^3 \), then the traces on \( \partial \mathcal{M} \) of \( \xi, \zeta, \eta, k, l, m \) make sense and belong for each side of \( \mathcal{M} \) to either \( H^{-1}(0, L_1) \) or \( H^{-1}(0, L_2) \); furthermore, these trace mappings are continuous from \( D(A) \) endowed with the norm \((|u|_{L^2(\mathcal{M})^3}^2 + |Au|_{L^2(\mathcal{M})^3}^2)^{1/2} \) into the corresponding space \( H^{-1} \). Hence the definition of \( D(A) \) is meaningful.

We consider two different adjoints of \( A \), namely the adjoint \( A^* \) of \( A \) in the classical sense (see [8, 11]) and its restriction \( A^*_{\mathcal{M}} \) to the closure of the smooth functions (see below). Their domains are defined as follows:
\[ D(A^*) = \{\tilde{u} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in H \text{ such that } A\tilde{u} \in H \text{ and } \tilde{\zeta}(L_1, y) = \tilde{\zeta}(L_1, y) \]
\[ = \tilde{\eta}(0, y) = 0, \tilde{k}(x, L_2) = \tilde{m}(x, L_2) = \tilde{\zeta}(x, 0) = 0 \}, \] (3.9)
\[ D(A^*_{\mathcal{M}}) = \text{Closure in } D(A^*) \text{ of } D(A^*) \cap C^3(\mathcal{M})^3. \] (3.10)

Here
\[
\tilde{k} = k(\tilde{u}) = \tilde{v}_0 \tilde{\zeta} + \Phi_1 \tilde{\zeta}, \quad \tilde{l} = l(\tilde{u}) = \tilde{v}_0 \tilde{\zeta} + \Phi_1 (\tilde{\zeta} - \tilde{\eta}),
\]
\[
\tilde{m} = m(\tilde{u}) = \tilde{v}_0 \tilde{\eta} - \Phi_1 \tilde{\zeta}.
\] (3.11)

For \( \tilde{u} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}) \in D(A^*) \), we define
\[ A^* \tilde{u} = \left( \begin{array}{c}
A_{11}^* \tilde{u} \\
A_{22}^* \tilde{u} \\
A_{33}^* \tilde{u}
\end{array} \right) = \left( \begin{array}{c}
-\tilde{u}_0 \frac{\partial \tilde{\eta}}{\partial x} - \frac{\partial \tilde{k}}{\partial y} + \tilde{f} \tilde{\zeta}, \\
-\tilde{u}_0 \frac{\partial \tilde{\eta}}{\partial x} - \frac{\partial \tilde{l}}{\partial y} - \tilde{f} (\tilde{\zeta} + \tilde{\eta}), \\
-\tilde{u}_0 \frac{\partial \tilde{\eta}}{\partial x} - \frac{\partial \tilde{m}}{\partial y} + \tilde{f} \tilde{\zeta}
\end{array} \right). \] (3.12)
When \( A \) and \( A^* \) are restricted to smooth functions, one can easily check that

\[
(Au, \tilde{u})_H = (u, A^*_x \tilde{u})_H = -(u, A\tilde{u})_H, \\
(Au, u)_H \geq 0, \quad (\tilde{u}, A^*_x \tilde{u})_H \geq 0,
\]

i.e., for every \( u \in D(A) \cap C^1(\bar{\mathcal{M}})^3 \), \( \tilde{u} \in D(A^*_x) \); however these results are not known in the general case when \( u \in D(A) \) and \( \tilde{u} \in D(A^*_x) \). Because of this limitation, one cannot apply the Hill–Yoshida theorem as in the problem of Section 2. Nevertheless, one can show that if

\[
f \in L^\infty(0, T; H), \quad f' \in L^1(0, T; H), \quad u_0 \in D(A)
\]

then there exists at least one solution \( u \) of (3.4) satisfying

\[
u \in L^\infty(0, T; D(A)), \quad \frac{du}{dt} \in L^\infty(0, T; H).
\]

**Remark 3.1.** As we said, the uniqueness of this \( u \) is related to the positivity of \( A \) \((Au, u)_H \geq 0, \forall u \in D(A)\) which is not known. For this reason, we will only obtain in Theorem 4.2 the convergence of subsequences \( h', k' \to 0 \). Note also that our approximation procedure below is constructive, that is it shows the existence of an \( u \) satisfying (3.4), (3.6).

### 3.1. FV Discretization

We start first by introducing the 2D structured FV mesh of \( \mathcal{M} = (0, L_1) \times (0, L_2) \) that we are using throughout this section. We obtain the FV mesh by choosing the meshes \( h_x^1, \ldots, h_x^M, h_y^1, \ldots, h_y^N \), and then the points \((x_1^1, y_1^1), \ldots, (x_{M+1}^1, y_{N+1}^N)\) so that:

\[
x_1^2 = 0, \quad x_3^2 = h_x^1, \quad x_5^2 = h_x^1 + h_x^2, \ldots, x_{M+1}^2 = L_1 = \sum_{i=1}^{M} h_x^i,
\]

\[
y_1^2 = 0, \quad y_3^2 = h_y^1, \quad y_5^2 = h_y^1 + h_y^2, \ldots, y_{N+1}^2 = L_2 = \sum_{i=1}^{N} h_y^i.
\]

The finite volumes (cells) are the rectangles \( K_{ij} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \), \( 1 \leq i \leq M, 1 \leq j \leq N \). We then define the \((x_i, y_j)\)'s that are the centers of the cells \( K_{ij}, 1 \leq i \leq M, 1 \leq j \leq N \).
We now discretize equations (3.1) by finite volumes. We integrate the first equation in (3.1) over the control volume \(K_{ij}\) and divide by its volume \(h_i^x h_j^y\); we find

\[
\frac{d\xi}{dt} + \frac{\bar{u}_0}{h_i^x h_j^y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[ \xi(x_{i+\frac{1}{2}}, y, t) - \xi(x_{i-\frac{1}{2}}, y, t) \right] dy
\]

\[
+ \frac{1}{h_i^x h_j^y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[ k(x, y_{j+\frac{1}{2}}, t) - k(x, y_{j-\frac{1}{2}}, t) \right] dx - \bar{f}\xi = f_{ij}.
\]

Equivalently

\[
\frac{d\tilde{\xi}_{ij}}{dt} + \frac{\bar{u}_0}{h_i^x h_j^y} \left( F_{i+\frac{1}{2}j}^\xi - F_{i-\frac{1}{2}j}^\xi \right) + \frac{1}{h_i^x h_j^y} \left( F_{y+\frac{1}{2}i}^k - F_{y-\frac{1}{2}i}^k \right) - \bar{f}\xi = f_{ij}, \tag{3.16}
\]

where the fluxes are \(F_{i+\frac{1}{2}j}^\xi = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \xi(x_{i+\frac{1}{2}}, y, t) dy\) and \(F_{y+\frac{1}{2}i}^k = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} k(x, y_{j+\frac{1}{2}}, t) dy\). These fluxes are approximated as follows

\[
F_{i+\frac{1}{2}j}^\xi = h_j^y \left[ \gamma_{i+\frac{1}{2}}(\xi_{i+1j} - \xi_{ij}) + (1 - \gamma_{i+\frac{1}{2}})(\xi_{ij} - \xi_{i-1j}) \right],
\]

\[
F_{y+\frac{1}{2}i}^k = h_i^x \left[ \gamma_{y+\frac{1}{2}}(k_{ij+1} - k_{ij}) + (1 - \gamma_{y+\frac{1}{2}})(k_{ij} - k_{ij-1}) \right], \tag{3.17}
\]

with \(\gamma_{i+\frac{1}{2}}, \gamma_{y+\frac{1}{2}} \in [0, 1]\).

We define

\[
D^x_h \xi_h = \frac{\gamma_{i+\frac{1}{2}}(\xi_{i+1j} - \xi_{ij}) + (1 - \gamma_{i+\frac{1}{2}})(\xi_{ij} - \xi_{i-1j})}{h_i^x} \quad \text{on } K_{ij},
\]

\[
D^y_h k_h = \frac{\gamma_{y+\frac{1}{2}}(k_{ij+1} - k_{ij}) + (1 - \gamma_{y+\frac{1}{2}})(k_{ij} - k_{ij-1})}{h_j^y} \quad \text{on } K_{ij}. \tag{3.18}
\]

Here we have defined for \(u_h = (\xi_h, \zeta_h, \eta_h)\) the associate discrete functions \(k_h, l_h, m_h\) by

\[
k_h = \tilde{u}_0 \xi_h + \Phi_1 \zeta_h, \quad l_h = \tilde{u}_0 \zeta_h + \Phi_1 (\xi_h - \eta_h), \quad m_h = \tilde{u}_0 \eta_h - \Phi_1 \zeta_h. \tag{3.19}
\]

Thus the scheme (3.16) can be written as

\[
\frac{d\tilde{\xi}_{ij}}{dt} + \tilde{u}_0 (D^x_h \xi_h)_{|K_{ij}} + (D^y_h k_h)_{|K_{ij}} - \bar{f}\xi = f_{ij}, \tag{3.20}
\]
Discretizing similarly the second and third equations in (3.1), we find the FV scheme for (3.1) in the form:

\[
\begin{align*}
\frac{d\tilde{\zeta}_{ij}}{dt} & + \tilde{u}_0(D_h^{x}\tilde{\zeta}_{i\hat{k}_{ij}})|_{K_{ij}} + (D_h^{x}k_h)|_{K_{ij}} - \tilde{f}\tilde{\zeta}_{ij} = f_{1ij}, \\
\frac{d\tilde{\eta}_{ij}}{dt} & + \tilde{u}_0(D_h^{x}\tilde{\eta}_{i\hat{k}_{ij}})|_{K_{ij}} + (D_h^{x}l_h)|_{K_{ij}} + \tilde{f}(\tilde{\zeta}_{ij} + \eta_{ij}) = f_{2ij}, \\
\frac{d\tilde{\eta}_{ij}}{dt} & + \tilde{u}_0(D_h^{x}\eta_{i\hat{k}_{ij}})|_{K_{ij}} + (D_h^{x}m_h)|_{K_{ij}} - \tilde{f}\tilde{\zeta}_{ij} = f_{3ij}.
\end{align*}
\]

We enforce the boundary conditions in (3.7) by setting

\[
\begin{align*}
\tilde{\zeta}_{0j} &= -\tilde{\zeta}_{1j}, \quad \tilde{\zeta}_{M+1j} = \tilde{\zeta}_{Mj}, \quad \tilde{\zeta}_{0j} = -\tilde{\zeta}_{1j}, \quad \tilde{\zeta}_{M+1j} = \tilde{\zeta}_{Mj}; \\
\eta_{0j} &= \eta_{1j}, \quad \eta_{M+1j} = -\eta_{Mj}, \\
k_{i0} &= -k_{i1}, \quad k_{iN+1} = k_{iN}; \\
m_{0i} &= -m_{i1}, \quad m_{iN+1} = m_{iN}; \\
\tilde{\zeta}_{i0} &= \tilde{\zeta}_{i1}, \quad \tilde{\zeta}_{iN+1} = -\tilde{\zeta}_{iN},
\end{align*}
\]

for \(1 \leq i \leq M, 1 \leq j \leq N\).

To actually implement the scheme (3.21) and its time discretized version, we now introduce the following discrete function spaces

\[
\begin{align*}
\mathbf{V}_h &= \text{Space of triples } (\tilde{\zeta}_h, \tilde{\eta}_h, \eta_h) \text{ where } \tilde{\zeta}_h, \tilde{\eta}_h, \eta_h \text{ are steps function on } K_{ij}, \\
& \quad i = 1, \ldots, M, j = 1, \ldots, N, \\
\mathbf{V}_{0h} &= \text{Space of triples } (\tilde{\zeta}_h, \tilde{\eta}_h, \eta_h) \text{ where } \tilde{\zeta}_h, \tilde{\eta}_h, \eta_h \text{ are steps function on } K_{ij}, \\
& \quad i = 0, \ldots, M+1, j = 0, \ldots, N+1, (i, j) \notin \{(0,0), (0, N+1), (M+1, N+1), (M+1, 0)\} \text{ with the boundary conditions (3.22).} \\
\mathbf{V}_{1h} &= \text{Space of triples } (\tilde{\zeta}_h, \tilde{\eta}_h, \eta_h) \text{ where } \tilde{\zeta}_h, \tilde{\eta}_h, \eta_h \text{ are steps function on } K_{ij}, \\
& \quad i = 0, \ldots, M+1, j = 0, \ldots, N+1, (i, j) \notin \{(0,0), (0, N+1), (M+1, N+1), (M+1, 0)\} \text{ with the boundary conditions (3.23) below:} \\
& \quad \begin{align*}
\tilde{\zeta}_{0j} &= \tilde{\zeta}_{1j}, \quad \tilde{\zeta}_{M+1j} = -\tilde{\zeta}_{Mj}; \\
\tilde{\zeta}_{i0} &= \tilde{\zeta}_{i1}, \quad \tilde{\zeta}_{iN+1} = -\tilde{\zeta}_{iN}; \\
\tilde{\eta}_{0j} &= -\tilde{\eta}_{1j}, \quad \tilde{\eta}_{M+1j} = \tilde{\eta}_{Mj}, \\
\tilde{k}_{i0} &= \tilde{k}_{i1}, \quad \tilde{k}_{iN+1} = -\tilde{k}_{iN}; \\
\tilde{m}_{i0} &= \tilde{m}_{i1}, \quad \tilde{m}_{iN+1} = -\tilde{m}_{iN}; \\
\end{align*}
\]

**Remark 3.2.** Like in the 1D case, the support of the functions in \(\mathbf{V}_h\) is \(\mathcal{M}\) while the support of the functions in \(\mathbf{V}_{0h}\) and \(\mathbf{V}_{1h}\) is larger than \(\mathcal{M}\). In what

\[1\text{Like in the continuous case, we use the notations } k_h = \tilde{v}_0\tilde{\zeta}_h + \Phi_1\tilde{\zeta}_h, \quad l_h = \tilde{v}_0\tilde{\zeta}_h + \Phi_1(\tilde{\zeta}_h - \eta_h), \\
m_h = \tilde{v}_0\eta_h - \Phi_1\tilde{\zeta}_h, \quad \text{and } \tilde{\zeta}_h = \tilde{v}_0\tilde{\zeta}_h + \Phi_1\tilde{\zeta}_h, \quad \tilde{\eta}_h = \tilde{v}_0\tilde{\zeta}_h + \Phi_1(\tilde{\zeta}_h - \eta_h), \quad \tilde{l}_h = \tilde{v}_0\tilde{\zeta}_h + \Phi_1(\tilde{\zeta}_h - \eta_h), \quad \tilde{m}_h = \tilde{v}_0\eta_h - \Phi_1\tilde{\zeta}_h.\]
follows, if the supports of the functions are larger than $\mathcal{M}$, then their scalar products or norms in $L^2(\mathcal{M})$ are the scalar products or norms of their restrictions to $\mathcal{M}$.

Scalar product and norm on $\mathbf{V}_h$: For $\mathbf{u}_h = (\xi_h, \zeta_h, \eta_h)$, $\tilde{\mathbf{u}}_h = (\tilde{\xi}_h, \tilde{\zeta}_h, \tilde{\eta}_h) \in \mathbf{V}_h$, we define

$$
(u_h, \tilde{u}_h)_h = (\xi_h, \tilde{\xi}_h)_H + (\zeta_h, \tilde{\zeta}_h)_H + (\eta_h, \tilde{\eta}_h)_H,
$$

$$
|\mathbf{u}_h|_h = (\mathbf{u}_h, \mathbf{u}_h)_h^{\frac{1}{2}} = \left( \sum_{i=1}^{M} \sum_{j=1}^{N} (\xi_{ij}^2 + \zeta_{ij}^2 + \eta_{ij}^2) h_i^x h_j^y \right)^{\frac{1}{2}}.
$$

Scalar products and norms on $\mathbf{V}_{0h}$, $\mathbf{V}_{1h}$: Let us first define the following discrete FV partial derivative for a step function defined on the $K_{ij}$, $u_h = \sum_{i=1}^{M} \sum_{j=1}^{N} u_{ij} \chi_{K_{ij}}$,

$$
\nabla^x_h u_h := \frac{u_{i+1,j} - u_{ij}}{h_{i+\frac{1}{2}}^x} \quad \text{on } K_{i+\frac{1}{2},j}, \quad 0 \leq i \leq M, \quad 1 \leq j \leq N, \quad (3.24)
$$

$$
\nabla^y_h u_h := \frac{u_{i,j+1} - u_{ij}}{h_{j+\frac{1}{2}}^y} \quad \text{on } K_{i+\frac{1}{2},j}, \quad 1 \leq i \leq M, \quad 0 \leq j \leq N.
$$

Here

$$
h_{i+\frac{1}{2}}^x := h_1^x, \quad h_{i-\frac{1}{2}}^x := x_{i+1} - x_i, \quad 1 \leq i \leq M - 1, \quad h_{M+\frac{1}{2}}^x = h_M^x,
$$

$$
h_{j+\frac{1}{2}}^y := h_1^y, \quad h_{j-\frac{1}{2}}^y := y_{j+1} - y_j, \quad 1 \leq j \leq N - 1, \quad h_{j+\frac{1}{2}}^y = h_j^y.
$$

and

$$
K_{i+\frac{1}{2},j} = (x_i, x_{i+1}) \times (y_j, y_{j+\frac{1}{2}}), \quad 0 \leq i \leq M, \quad 1 \leq j \leq N, \quad (3.26)
$$

$$
K_{i+\frac{1}{2},j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_j, y_{j+1}), \quad 1 \leq i \leq M, \quad 0 \leq j \leq N.
$$

Now we define the following scalar product and norm for $\mathbf{u}_h = (\xi_h, \zeta_h, \eta_h)$, $\tilde{\mathbf{u}}_h = (\tilde{\xi}_h, \tilde{\zeta}_h, \tilde{\eta}_h) \in \mathbf{V}_{0h}$ (or $\mathbf{V}_{1h}$)

$$
((\mathbf{u}_h, \tilde{\mathbf{u}}_h))_h = (\nabla^x_h \xi_h, \nabla^x_h \tilde{\xi}_h)_h + (\nabla^x_h \zeta_h, \nabla^x_h \tilde{\zeta}_h)_h + (\nabla^x_h \eta_h, \nabla^x_h \tilde{\eta}_h)_h
$$

$$
+ (\nabla^y_h \xi_h, \nabla^y_h \tilde{\xi}_h)_h + (\nabla^y_h \zeta_h, \nabla^y_h \tilde{\zeta}_h)_h + (\nabla^y_h \eta_h, \nabla^y_h \tilde{\eta}_h)_h, \quad (3.27)
$$

$$
\|\mathbf{u}_h\|_h = ((\mathbf{u}_h, \mathbf{u}_h))_h^{\frac{1}{2}}. \quad (3.28)
$$
We then define the operator $A_h : \mathbf{V}_{0h} \to \mathbf{V}_h$ by $A_h \mathbf{u}_h = (A_{1h} \mathbf{u}_h, A_{2h} \mathbf{u}_h, A_{3h} \mathbf{u}_h)$ where

$$
\begin{align*}
A_{1h} \mathbf{u}_h &= \bar{u}_0 D^x_\xi \xi_h + D^x_\xi y_h - \bar{f} \xi_h, \\
A_{2h} \mathbf{u}_h &= \bar{u}_0 D^x_\xi \eta_h + D^x_\xi y_h + \bar{f} (\xi_h + \eta_h), \\
A_{3h} \mathbf{u}_h &= \mathbf{u}_0 D^x_\eta \eta_h + D^x_\eta m_h - \bar{f} \xi_h,
\end{align*}
$$

and $D^x_\xi, D^x_\eta$ are as in (3.18). Then the scheme (3.21) can be written as

$$
\begin{align*}
\frac{d\xi_h}{dt} + A_{1h} \mathbf{u}_h &= f_{1h}, \\
\frac{d\eta_h}{dt} + A_{2h} \mathbf{u}_h &= f_{2h}, \\
\frac{d\eta_h}{dt} + A_{3h} \mathbf{u}_h &= f_{3h},
\end{align*}
$$

or

$$
\frac{d\mathbf{u}_h}{dt} + A_h \mathbf{u}_h = f_h. \tag{3.30}
$$

The following lemma shows the positiveness of the operator $A_h$:

**Lemma 3.3.** Under the assumptions (3.2), $\gamma_{i+\frac{1}{2}} = 1/2$, $0 \leq i \leq N$, $1 \leq j \leq N$ and $\gamma_{ij+\frac{1}{2}} = 1/2$, $1 \leq M$, $0 \leq j \leq N$, we have $(A_h \mathbf{u}_h, \mathbf{u}_h)_h \geq 0$ for $\mathbf{u}_h \in \mathbf{V}_h$.

**Proof.** We write

$$(A_h \mathbf{u}_h, \mathbf{u}_h)_h = (A_{1h} \mathbf{u}_h, \xi_h)_h + (A_{2h} \mathbf{u}_h, \xi_h)_h + (A_{3h} \mathbf{u}_h, \eta_h)_h$$

$$= \bar{u}_0 (D^x_\xi \xi_h + D^x_\xi y_h - \bar{f} \xi_h, \xi_h)_h + \bar{u}_0 (D^x_\xi \xi_h + D^x_\xi y_h + \bar{f} (\xi_h + \eta_h), \xi_h)_h$$

$$+ (\mathbf{u}_0 D^x_\eta \eta_h + D^x_\eta m_h - \bar{f} \xi_h, \eta_h)_h$$

$$= \bar{u}_0 (D^x_\xi \xi_h, \xi_h)_h + \bar{u}_0 (D^x_\xi \xi_h, \xi_h)_h + \mathbf{u}_0 (D^x_\eta \eta_h, \eta_h)_h$$

$$+ (D^x_\xi y_h, \xi_h)_h + (D^x_\eta m_h, \eta_h)_h$$

$$+ \bar{f} \left\{ -(\xi_h, \xi_h)_h + (\xi_h, \xi_h)_h + (\eta_h, \xi_h)_h - (\xi_h, \eta_h)_h \right\}. $$

By the symmetry of the scalar product, the coefficient of $\bar{f}$ is zero. By the definitions of $k_h$ and $m_h$, we find

$$(A_h \mathbf{u}_h, \mathbf{u}_h)_h = \bar{u}_0 (D^x_\xi \xi_h, \xi_h)_h + \bar{u}_0 (D^x_\xi \xi_h, \xi_h)_h + \mathbf{u}_0 (D^x_\eta \eta_h, \eta_h)_h$$

$$+ \left( D^x_\xi k_h, \frac{1}{v_0} k_h - \frac{\Phi_1}{v_0} \xi_h \right)_h + (D^x_\xi l_h, \xi_h)_h + \left( D^x_\eta m_h, \frac{1}{v_0} m_h + \frac{\Phi_1}{v_0} \xi_h \right)_h$$

$$+ \left( D^x_\eta m_h, \frac{\Phi_1}{v_0} \xi_h \right)_h. $$


By the definitions of $k_h$, $l_h$, $m_h$, we also find
\[
l_h + \frac{\Phi_1}{\tilde{v}_0}(m_h - k_h) = \tilde{v}_0\varphi_h^* + \Phi_1(\varphi_h - \eta_h) + \frac{\Phi_1}{\tilde{v}_0}\{\tilde{v}_0\eta_h - \Phi_1\varphi_h - (\tilde{v}_0\varphi_h^* + \Phi_1\varphi_h^*)\}
\]
\[
= \left(\frac{\tilde{v}_0 - 2\Phi_1^2}{\tilde{v}_0}\right)\varphi_h.
\]
Thus we conclude that
\[
(A_h\varphi_h, \varphi_h)_h = \tilde{v}_0(\varphi_h, \varphi_h)_h + \tilde{v}_0(\varphi_h, \varphi_h)_h + \tilde{v}_0(\varphi_h, \varphi_h)_h + \left(\tilde{v}_0 - 2\Phi_1^2\right)(\varphi_h, \varphi_h)_h.
\]
It follows from Lemma 3.5 below that
\[
(\varphi_h, \varphi_h)_h, (\varphi_h, \varphi_h)_h, (\varphi_h, \varphi_h)_h, (\varphi_h, \varphi_h)_h \geq 0,
\]
\[
(\varphi_h, \varphi_h)_h, (\varphi_h, \varphi_h)_h \leq 0.
\]
Then because $u_0$ and $(\tilde{v}_0 - 2\Phi_1^2)$ are nonpositive, we conclude that $(A_h\varphi_h, \varphi_h)_h \geq 0$.  

**Remark 3.4.** We note that to ensure the positivity of the operator $A_h$, the boundary conditions of $\varphi_h$, $\varphi_h$ require that $\gamma_{i+\frac{1}{2},j} \leq 1/2$ while the boundary conditions of $\eta_h$ require that $\gamma_{i+\frac{1}{2},j} \geq 1/2$ (similarly for the $y$ direction, we should have $\gamma_{i,j+\frac{1}{2}} \leq 1/2$ for $k_h$, $m_h$ and $\gamma_{i,j+\frac{1}{2}} \geq 1/2$ for $\eta_h$). For simplicity, we choose the centered scheme, which means $\gamma_{i+\frac{1}{2},j} = 1/2$, $\gamma_{i,j+\frac{1}{2}} = 1/2$, which guarantees both requirements. More general cases will be considered elsewhere ($\gamma$ different for different directions and for different variables $\varphi, \varphi, \kappa, \eta, l, m$).

What is left in the proof of the above lemma is the following lemma:

**Lemma 3.5.** Let $\varphi_h$ be a step function defined on the $K_{ij}$'s, $0 \leq i \leq M + 1$, $0 \leq j \leq N + 1$, and $(i, j) \notin \{(0,0), (0, N + 1), (M + 1, N + 1), (M + 1, 0)\}$. We also suppose that $\gamma_{i+\frac{1}{2},j} = \gamma_{i,j+\frac{1}{2}} = 1/2$ for all $i, j$. Then
\[
\begin{aligned}
(D_h^x \varphi_h, \varphi_h)_h &\geq 0 \text{ if } \varphi_{ij} = -\varphi_{ij} \text{ and } \varphi_{M+1j} = \varphi_{Mj}, \quad 1 \leq j \leq N, \\
(D_h^x \varphi_h, \varphi_h)_h &\leq 0 \text{ if } \varphi_{ij} = \varphi_{ij} \text{ and } \varphi_{M+1j} = -\varphi_{Mj}, \quad 1 \leq j \leq N.
\end{aligned}
\]
(3.31)
And similarly
\[ \begin{cases} 
(D^y_h v_h, v_h)_h \geq 0 & \text{if } v_0 = -v_1 \text{ and } v_{N+1} = v_N, \ 1 \leq i \leq M, \\
(D^y_h v_h, v_h)_h \leq 0 & \text{if } v_0 = v_1 \text{ and } v_{N+1} = -v_N, \ 1 \leq i \leq M. 
\end{cases} \] (3.32)

**Proof.** We prove (3.31). The proof of (3.32) is similar. Because \( \gamma_{i+1/2} = \gamma_{i+1} = 1/2 \) for all \( i, j \), we have

\[
(D^x_h v_h, \nu_h)_h \\
= \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{N} \left[ (v_{i+1} - v_j) + (v_{j+1} - v_{i-1}) \right] v_j h^y_j \\
= \frac{1}{2} \sum_{j=1}^{N} \left[ \sum_{i=1}^{M} \left[ (v_{i+1} - v_j)^2 - (v_{i+1} - v_j)^2 + (v_j)^2 - (v_{i-1})^2 + (v_j - v_{i-1})^2 \right] \right] h^y_j \\
= \frac{1}{2} \sum_{j=1}^{N} \left\{ \sum_{i=1}^{M} \left[ (v_{i+1} - v_j)^2 - (v_{i+1} - v_j)^2 + (v_j - v_{i-1})^2 \right] \right\} h^y_j \\
= \frac{1}{2} \sum_{j=1}^{N} \left\{ (v_{M+1}^2 + (v_{Mj})^2 - (v_{M+1} - v_{Mj})^2 - (v_{Mj})^2 - (v_{Mj} - v_{Mj})) \right\} h^y_j \\
= \begin{cases} 
\frac{1}{2} \sum_{j=1}^{N} \left\{ (v_{Mj})^2 \right\} & \text{if } v_{Mj} = -v_{Mj} \text{ and } v_{M+1} = v_{Mj}, \ 1 \leq j \leq N, \\
-\frac{1}{2} \sum_{j=1}^{N} \left\{ (v_{Mj})^2 \right\} & \text{if } v_{Mj} = v_{Mj} \text{ and } v_{M+1} = -v_{Mj}, \ 1 \leq j \leq N. 
\end{cases}
\]

Hence (3.31) holds. \( \square \)

### 4. IMPLICIT EULER SCHEME FOR SW: STABILITY AND CONVERGENCE

We consider the following implicit Euler FV scheme for (3.30): **to find recursively the** \( u^n_h \) **such that** \( u^n_h = r_h u^n_0 := \frac{1}{h^y_j h^x_k} \int_{T^j_k} u^n_0 \) **on** \( K_j \) **and**

\[
\frac{u^n_h - u^{n-1}_h}{\Delta t} + A_h u^n_h = f^n_h, \ 1 \leq n \leq N. \] (4.1)
In this part, for the sake of simplicity, we assume that the constants $\gamma_{i+\frac{1}{2},j}$, $\gamma_{j+\frac{1}{2}}$ defined in (3.17) are all equal to 1/2. Furthermore, we assume that the mesh is uniform, that is, $h_x^i = h$, $h_y^j = h$ for $1 \leq i \leq M$, $1 \leq j \leq N$.

4.1. Stability Results

From Lemma 3.3, we see that the operator $A_h$ is positive. With the same proof as in Section 2 for Lemma 2.3, the stability lemma, the following stability results also hold for the solution of (4.1):

Lemma 4.1. If $\Delta t \leq x^*$ where $x^* (\approx 0.7968)$ is the positive root of $f(x) = e^{-2x} + x - 1$, the solutions $u^n_h$ to the scheme (4.1) satisfy, for all $1 \leq n \leq N$,

\[
|u^n_h|_{h}^2 \leq e^{2T} \left( |r_h u_0|_{h}^2 + \sup_{m=1,\ldots,N} |f^m_h|_{h}^2 \right),
\]

(4.2)

\[
|A_h u^n_h|_{h} \leq |A_h(r_h u_0)|_{h} + \sum_{m=2}^{n} |f^m_h - f^{m-1}_h|_{h} + |f^1_h|_{h}.
\]

(4.3)

4.2. Convergence Results

We consider the following approximate functions

\[
\begin{align*}
\bar{u}_{hk}(t) &:= u^{n-1}_h \quad \text{for } (n-1)\Delta t < t < n\Delta t, \quad n = 1, \ldots, N, \\
\bar{u}_{hk}(t) &:= u^n_h \quad \text{for } (n-1)\Delta t < t < n\Delta t, \quad n = 1, \ldots, N,
\end{align*}
\]

(4.4)

where the $u^n_h$ are the solutions of the problem (3.30). We then have the following convergence results for $\bar{u}_{hk}$:

Theorem 4.2. There exists a subsequence $\bar{u}_{h'k'}$ of the approximate solutions $\bar{u}_{hk}$, such that, as $h' \to 0, k' \to 0$,

\[
\begin{align*}
\bar{u}_{h'k'} &\text{ converges to } u \text{ in } L^\infty(0, T; H) \text{ weak-star,} \\
\frac{\bar{u}_{h'k'} - \bar{u}_{k'}}{k'} &\text{ converges to } \frac{d u}{d t} \text{ in } L^\infty(0, T; H) \text{ weak-star,}
\end{align*}
\]

where $u$ satisfies

\[
\begin{align*}
\frac{d}{d t} (u, \tilde{u})_H + (u, A^*_\tilde{u})_H &= (f, \tilde{u})_H, \quad \forall \tilde{u} \in D(A^*_\tilde{u}), \\
u(x, y, 0) &= u_0(x, y).
\end{align*}
\]

(4.5)
Furthermore, there exists \( M > 0 \) depending only on the data, such that
\[
|\mathbf{u}|_{L^\infty(0,T;\mathbf{H})} \leq M_\alpha^T \left( \|\mathbf{u}_0\|_\mathbf{H} + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{H})} \right),
\]
\[
|A\mathbf{u}|_{L^\infty(0,T;\mathbf{H})} \leq M \left( \|A\mathbf{u}_0\|_\mathbf{H} + \|\mathbf{f}'\|_{L^1(0,T;\mathbf{H})} + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{H})} \right).
\]

Before giving the proof of Theorem 4.2 above, we need the following discrete integration by parts lemmas:

**Lemma 4.3.** Let \( u_h, v_h \) be two step functions defined on the \( K_{ij} \), \( 0 \leq i \leq M + 1, \ 0 \leq j \leq N + 1, \) and \((i,j) \notin \{(0,0), (0,N+1), (M+1,N+1), (M+1,0)\} \). Suppose that either
\[
\begin{align*}
   u_{0j} = & -u_{1j}, \ u_{M+1j} = u_{Mj} \quad \text{and} \quad v_{0j} = v_{1j}, \ v_{M+1j} = -v_{Mj}, \\
   \text{or} \quad u_{0j} = & u_{1j}, \ u_{M+1j} = -u_{Mj} \quad \text{and} \quad v_{0j} = -v_{1j}, \ v_{M+1j} = v_{Mj},
\end{align*}
\]
for \( 1 \leq j \leq N \). Then we have
\[
\int_{\mathcal{R}} (D^x_h u_h) v_h \, dx \, dy = -\int_{\mathcal{R}} \hat{u}_h^x \nabla^x_h v_h \, dx \, dy. \tag{4.7}
\]

Similarly, if either
\[
\begin{align*}
   u_{j0} = & -u_{j1}, \ u_{jN+1} = u_{jN} \quad \text{and} \quad v_{j0} = v_{j1}, \ v_{jM+1} = -v_{Mj}, \\
   \text{or} \quad u_{j0} = & u_{j1}, \ u_{jN+1} = -u_{jN} \quad \text{and} \quad v_{j0} = -v_{j1}, \ v_{jN+1} = v_{jN},
\end{align*}
\]
then we have
\[
\int_{\mathcal{R}} (D^y_h u_h) v_h \, dx \, dy = -\int_{\mathcal{R}} \hat{u}_h^y \nabla^y_h v_h \, dx \, dy, \tag{4.9}
\]

where \( D^x_h, D^y_h \) are as in (3.18), \( \nabla^x_h, \nabla^y_h \) are as in (3.24) and
\[
\begin{align*}
   \hat{u}_h^x = & \frac{u_{i+1j} + u_{ij}}{2} \quad \text{on} \ K_{i+\frac{1}{2}j}, \ 0 \leq i \leq M, \ 1 \leq j \leq N, \\
   \hat{u}_h^y = & \frac{u_{ij+1} + u_{ij}}{2} \quad \text{on} \ K_{ij+\frac{1}{2}}, \ 1 \leq i \leq M, \ 0 \leq j \leq N.
\end{align*}
\]

**Proof.** We prove (4.7); the proof of (4.9) is similar. We start from the left-hand side of (4.7). Using \( \gamma_{i+\frac{1}{2}j} = \gamma_{ij+\frac{1}{2}} = 1/2 \), we have
\[
\begin{align*}
   \int_{\mathcal{R}} D^x_h u_h v_h \, dx \, dy = & \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{1}{2h_i^x} \int_{K_{ij}} \left[ u_{i+1j} + u_{ij} - (u_{ij} + u_{i-1j}) \right] v_j \, dx \, dy \\
   = & \sum_{j=1}^{N} \frac{h_j^y}{2} \sum_{i=1}^{M} \left[ u_{i+1j} + u_{ij} - (u_{ij} + u_{i-1j}) \right] v_j.
\end{align*}
\]
Reordering the inner sum, we find

\[
\int_{\mathbb{R}^2} D_h^x u_h v_h \, dx \, dy
\]

\[
= \sum_{j=1}^{N} \frac{h_j^x}{2} \left[ \sum_{i=1}^{M-1} (u_{i+1j} + u_{ij})(v_{ij} - v_{i+1j}) - (u_{ij} + u_{0j})v_{ij} + (u_{M+1j} + u_{Mj})v_{Mj} \right].
\]

Because \( u_{0j} = -u_{ij} \) and \( v_{M+1j} = -v_{Mj} \), we have

\[
\int_{\mathbb{R}^2} D_h^x u_h v_h \, dx \, dy
\]

\[
= \sum_{j=1}^{N} \frac{h_j^x}{2} \left[ \sum_{i=1}^{M-1} (u_{i+1j} + u_{ij})(v_{ij} - v_{i+1j}) + (u_{M+1j} + u_{Mj})v_{Mj} \right]
\]

\[
= \sum_{j=1}^{N} \left[ \sum_{i=1}^{M-1} \frac{u_{i+1j} + u_{ij} - v_{i+1j} - u_{ij} + u_{M+1j} + u_{Mj} - 2v_{Mj} - 2u_{M+1j} - 2u_{Mj}}{2} \right] h_j^x h_{i+\frac{1}{2}}^y
\]

\[
= \sum_{j=1}^{N} \sum_{i=1}^{M-1} \left[ \int_{K_{i+\frac{1}{2},j}} \hat{u}_j^x \nabla_h^x v_h \, dx \, dy - \int_{K_{M+\frac{1}{2},j} \cap \mathbb{R}^2} \hat{u}_j^x \nabla_h^x v_h \, dx \, dy \right].
\]

Noting that \( \nabla_h^x v_h = (v_{ij} - v_{i+1j})/h_j^x = 0 \) on \( K_{i+\frac{1}{2},j} \), we conclude that

\[
\int_{\mathbb{R}^2} D_h^x u_h v_h \, dx \, dy = -\int_{\mathbb{R}^2} \hat{u}_j^x \nabla_h^x v_h \, dx \, dy.
\]

For the case \( u_{0j} = u_{ij} \), \( u_{M+1j} = -u_{Mj} \), and \( v_{0j} = -v_{ij} \), \( v_{M+1j} = v_{Mj} \), we note that \( \nabla_h^x v_h = 0 \) on \( K_{M+\frac{1}{2},j} \), the result also holds.

And now we have a discrete integration by parts formula for the operator \( A_h \):

**Lemma 4.4.** We assume that \( u_h = (\xi_h, \zeta_h, \eta_h) \in V_h \), \( \tilde{u}_h = (\tilde{\xi}_h, \tilde{\zeta}_h, \tilde{\eta}_h) \in V_{1h} \), and we define \( k_h, \ell_h, m_h \) and \( \tilde{k}_h, \tilde{\ell}_h, \tilde{m}_h \) as in (3.19) and (3.11). Then

\[
(A_h u_h, \tilde{u}_h)_h = -\tilde{u}_0(\tilde{\xi}_h, \nabla_h^x \tilde{\zeta}_h)_h - \tilde{u}_0(\tilde{\zeta}_h, \nabla_h^x \tilde{\eta}_h)_h - \tilde{u}_0(\tilde{\eta}_h, \nabla_h^x \tilde{\xi}_h)_h
\]

\[
- \tilde{\xi}_h(\xi_h, \nabla_h^x \zeta_h)_h - \tilde{\zeta}_h(\zeta_h, \nabla_h^x \ell_h)_h - \tilde{\eta}_h(\eta_h, \nabla_h^x m_h)_h
\]

\[
+ \tilde{f} \left[ (\xi_h, \zeta_h)_h - (\zeta_h, \ell_h + \zeta_h)_h + (\eta_h, \zeta_h)_h \right]. \tag{4.11}
\]
Proof. We write
\[ (A_h u_h, \bar{u}_h) = (A^1_h u_h, \tilde{z}_h) + (A^2_h u_h, \tilde{\zeta}_h) + (A^3_h u_h, \tilde{\eta}_h) \]
\[ = (\bar{u}_0 D^x_h \zeta_h + D^y_h k_h - \bar{f} \tilde{\zeta}_h, \tilde{\zeta}_h) + (\bar{u}_0 D^x_h \bar{\zeta}_h + D^y_h k_h + \bar{f} (\zeta_h + \eta_h), \tilde{\eta}_h) \]
\[ + (u_0 D^x_h \eta_h + D^y_h m_h - \bar{f} \tilde{\eta}_h, \tilde{\eta}_h) \]
\[ = I + II + III, \]
where
\[ I = (\bar{u}_0 D^x_h \zeta_h, \tilde{\zeta}_h) + (\bar{u}_0 D^x_h \bar{\zeta}_h, \tilde{\zeta}_h) + (u_0 D^x_h \eta_h, \tilde{\eta}_h), \]
\[ II = (D^x_h k_h, \tilde{\zeta}_h) + (D^y_h \bar{\zeta}_h, \tilde{\zeta}_h) + (D^y_h m_h, \tilde{\eta}_h), \]
\[ III = -(\bar{f} \tilde{\zeta}_h, \tilde{\eta}_h) + (\bar{f} (\zeta_h + \eta_h), \tilde{\eta}_h) - (\bar{f} \tilde{\eta}_h, \tilde{\eta}_h). \]

By Lemma 4.3, we find for I
\[ I = -\bar{u}_0 \tilde{\zeta}_h, \tilde{\eta}_h - \bar{u}_0 \tilde{\zeta}_h, \tilde{\eta}_h - u_0 \tilde{\eta}_h, \tilde{\eta}_h; \]
and for II with the use of (3.19) and (3.11), we see that
\[ II = -\tilde{y}_h, \tilde{y}_h - \tilde{\eta}_h, \tilde{\eta}_h - \tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h \]
\[ - \tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h - \tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h \]
\[ - \tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h - \tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h \]
\[ = -\tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h - \tilde{\eta}_h, \tilde{\eta}_h - \Phi_1 \tilde{\eta}_h, \tilde{\eta}_h. \]
Finally for III,
\[ III = \bar{f} \left\{ -\tilde{\eta}_h, \tilde{\eta}_h + (\tilde{\eta}_h, \tilde{\eta}_h) - (\tilde{\eta}_h, \tilde{\eta}_h) \right\} \]
\[ = \bar{f} \left\{ (\tilde{\eta}_h, \tilde{\eta}_h) - (\tilde{\eta}_h, \tilde{\eta}_h) + (\tilde{\eta}_h, \tilde{\eta}_h) \right\}. \]
Thus the lemma follows. \(\square\)

The following lemma shows the convergence of the functions that appear on the right-hand side of the discrete integration by parts (4.7) and (4.9).

Lemma 4.5. We assume that the mesh is uniform and \( \gamma_{i+\frac{1}{2},j} = \gamma_{j+\frac{1}{2}} = 1/2 \) for all \( i,j \). Then if \( u_h \to u \) in \( H \) weakly, then \( \hat{u}_h, \hat{u}_h^j \to u \) in \( H \) weakly.
Proof. We prove that \( \hat{u}_h^x \rightharpoonup u \) in \( H \) weakly. By definition

\[
\hat{u}_h^x = \frac{u_{i+1j} + u_{ij}}{2} \quad \text{on } K_{i+\frac{1}{2}j}^x,
\]

and

\[
u_h = \begin{cases} 
  u_{i+1j} & \text{on } K_{i+1j}^{x-} := K_{i+1j} \cap K_{i+\frac{1}{2}j}^x, \\
  u_{ij} & \text{on } K_{ij}^{x+} := K_{ij} \cap K_{i+\frac{1}{2}j}^x.
\end{cases}
\]

Thus

\[
\hat{u}_h^x - u_h = \begin{cases} 
  \frac{u_{i+1j} - u_{ij}}{2} & \text{on } K_{i+1j}^{x-}, \\
  -\frac{u_{i+1j} - u_{ij}}{2} & \text{on } K_{ij}^{x+}.
\end{cases}
\]

For \( \varphi \in \mathcal{D}(\mathcal{M}) \), we obtain

\[
\int_{\mathcal{M}} (\hat{u}_h^x - u_h) \varphi(x, y) \, dx \, dy = \sum_{j=1}^{N} \left[ \sum_{i=1}^{M} \int_{K_{ij}^{x+}} \frac{u_{i+1j} - u_{ij}}{2} \varphi(x, y) \, dx \, dy - \sum_{i=1}^{M} \int_{K_{i+1j}^{x-}} \frac{u_{i+1j} - u_{ij}}{2} \varphi(x, y) \, dx \, dy \right].
\]

Let us consider the first sum: for \( h \) sufficiently small so that \( \varphi_{|K_{ij}} = \varphi_{|K_{ij}^M} = 0 \), we find

\[
\sum_{i=1}^{M} \int_{K_{ij}^{x+}} (u_{i+1j} - u_{ij}) \varphi(x, y) \, dx \, dy = \sum_{i=1}^{M-1} (u_{i+1j} - u_{ij}) \int_{K_{ij}^{x+}} \varphi(x, y) \, dx \, dy
\]

\[
= \sum_{i=2}^{M-1} u_{ij} \left( \int_{K_{i-1j}^{x+}} \varphi(x, y) \, dx \, dy - \int_{K_{ij}^{x+}} \varphi(x, y) \, dx \, dy \right)
\]

\[
= \sum_{i=2}^{M-1} u_{ij} \int_{K_{ij}^{x+}} (\varphi(x-h^x, y) - \varphi(x, y)) \, dx \, dy.
\]

Because \( \varphi(x-h^x, y) - \varphi(x, y) = O(h^x) \) and \( |u_h|_{H}^2 = \sum_{i,j=1}^{M,N} (u_{ij})^2 h^x h^y \) is bounded, we easily see that

\[
\left| \sum_{i=1}^{M} \int_{K_{ij}^{x+}} (u_{i+1j} - u_{ij}) \varphi(x, y) \, dx \, dy \right| \to 0,
\]
as \( h^x, h^y \rightarrow 0 \). By the same calculations, the second sum also tends to zero as \( h^x, h^y \) go to zero. The lemma is proved. \( \Box \)

Now we are ready to prove the convergence results of Theorem 4.2.

**Proof of Theorem 4.2.** From (4.2) and (4.3), we infer that

\[
\|\tilde{u}_{h^k}\|_{L^\infty(0,T;\mathbf{H})} \leq Me^T (|u_0|_{\mathbf{H}} + \|f\|_{L^\infty(0,T;\mathbf{H})})^2, \\
\|A_h\tilde{u}_{h^k}\|_{L^\infty(0,T;\mathbf{H})} \leq M (|A(u_0)|_{\mathbf{H}} + \|f\|_{L^1(0,T;\mathbf{H})} + \|f\|_{L^\infty(0,T;\mathbf{H})}),
\]

where \( M > 0 \) depends only on \( u_0, f \). Thus there exist subsequences \( h' \)
\( k' \rightarrow 0 \) such that \( \tilde{u}_{h'/k'} \) converges to \( u \) and \( (\tilde{u}_{h'/k'} - u_{h'/k'})/(k') \) converges to \( du/dt \) in \( L^\infty(0,T;\mathbf{H}) \) weak-star as \( k', h' \rightarrow 0 \).

Fix \( \tilde{u} = (\tilde{\xi}, \tilde{\eta}, \tilde{\eta}) \in D(A_{x}^\circ) \) and \( \psi \in C([0,T]) \) such that \( \psi(T) = 0 \), we set \( \tilde{u}_{k'} = r_t \tilde{u} \).

Because \( \tilde{u} \) is smooth, by applications of Taylor's formula we can prove that, when \( h' \rightarrow 0 \),

\[
\begin{align*}
\tilde{u}_{h'/k'} & \rightarrow \tilde{u} \quad \text{strongly in } \mathbf{H}, \\
\nabla^x \tilde{u}_{k'} & \rightarrow (\frac{\nabla^x \tilde{\xi}}{h}, \frac{\nabla^y \tilde{\xi}}{h}, \frac{\nabla^z \tilde{\xi}}{h}) \quad \text{strongly in } \mathbf{H}, \\
\nabla^y \tilde{u}_{k'} & \rightarrow (\frac{\nabla^x \tilde{\eta}}{h}, \frac{\nabla^y \tilde{\eta}}{h}, \frac{\nabla^z \tilde{\eta}}{h}) \quad \text{strongly in } \mathbf{H}.
\end{align*}
\]

Taking the scalar product in \( \mathbf{H} \) of (3.30) with \( \tilde{u}_{k'} \triangle \psi^{n-1} \), and summing for \( n = 1, \ldots, N_t \), we obtain

\[
\sum_{n=1}^{N_t} (u^0_{k'}, \tilde{u}_k, \psi^{n-1}_{k'})_h + \Delta t \sum_{n=1}^{N_t} (A_hu^n_{k'}, \tilde{u}_k, \psi^{n-1}_{k'})_h = \Delta t \sum_{n=1}^{N_t} (f^n_{k'}, \tilde{u}_k, \psi^{n-1}_{k'})_h.
\]

Reordering the first sum and noting that \( \psi^{N_t} = \psi(T) = 0 \), we find

\[
\sum_{n=1}^{N_t} (u^0_{k'}, \tilde{u}_k, (\psi^{n-1} - \psi^n))_h + \Delta t \sum_{n=1}^{N_t} (A_hu^n_{k'}, \tilde{u}_k, \psi^{n-1})_h
\]

\[
= (u^0_{k'}, \tilde{u}_k)_h \psi(0) + \Delta t \sum_{n=1}^{N_t} (f^n_{k'}, \tilde{u}_k, \psi^{n-1})_h.
\]

We now write the first sum of (4.15) as

\[
\sum_{n=1}^{N_t} \left( u^n_{k'}, \tilde{u}_k, (\psi^{n-1} - \psi^n) \right)_h = \sum_{n=1}^{N_t} \int_{(n-1)\Delta t}^{n\Delta t} \left( \tilde{u}_{k'/k}, \tilde{u}_k, \frac{\psi_k(t) - \psi_k(t + k')}{k'} \right)_h dt
\]

\[
= \int_0^T \left( \tilde{u}_{k'/k}, \tilde{u}_k, \frac{\psi_k(t) - \psi_k(t + k')}{k'} \right)_h dt.
\]
By Lemma 4.4, we write the second term of (4.15) as

\[
\Delta t \sum_{n=1}^{N_t} \left( A_{H, H'} \begin{pmatrix} u_{H'}^n \cr \tilde{u}_{H'}^n \end{pmatrix}, \begin{pmatrix} \psi_{H'}^n \cr \tilde{\psi}_{H'}^n \end{pmatrix} \right)_{H} \psi^{n-1} = \Delta t \sum_{n=1}^{N_t} \left\{ -\tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\zeta}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\eta}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\kappa}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\zeta}_{H'}^n \right)_{H}
\right. \\
\left. - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\eta}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\kappa}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\zeta}_{H'}^n \right)_{H}
\right. \\
+ \tilde{f} \left[ \left( \tilde{\zeta}_{H'}, \tilde{\eta}_{H'} \right)_{H} - \left( \tilde{\kappa}_{H'}, \tilde{\zeta}_{H'} \right)_{H} \right] \psi^{n-1}
\right\} dt.
\]

Thus (4.15) becomes

\[
\int_0^T \left( \tilde{u}_{H'}, \tilde{u}_{H'} \begin{pmatrix} \psi_{H'}(t) - \psi_{H'}(t + k') \cr k' \end{pmatrix} \right)_{H} dt
\]

\[
+ \int_0^T \left\{ -\tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\zeta}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\zeta}_{H'}^n \right)_{H}
\right. \\
\left. - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\kappa}_{H'}^n \right)_{H} - \tilde{u}_0 \left( \tilde{\psi}_{H'}^n, \nabla_{H} \tilde{\zeta}_{H'}^n \right)_{H}
\right. \\
+ \tilde{f} \left[ \left( \tilde{\zeta}_{H'}, \tilde{\kappa}_{H'} \right)_{H} - \left( \tilde{\kappa}_{H'}, \tilde{\zeta}_{H'} \right)_{H} \right] \psi^{n-1}
\right\} dt
\]

\[
= \left( u_{H'}, \tilde{u}_{H'} \right)_{H} \psi(0) + \int_0^T \left( \tilde{f}_{H'}(t), \tilde{u}_{H'} \psi_{H'}(t) \right)_{H} dt.
\]

We now pass to the limit in (4.16). We know that \( \tilde{u}_{H'} \to \tilde{u} \) weakly in \( L^\infty(0, T; H) \), which means that \( \tilde{\zeta}_{H'} \to \tilde{\zeta}, \tilde{\kappa}_{H'} \to \tilde{\kappa} \) and \( \tilde{\eta}_{H'} \to \tilde{\eta} \) weakly in
By an application of Taylor’s formula, we conclude that 
\( \psi_{k'} \to \psi, (\psi_{k'}(\cdot + k') - \psi_{k'})/k' \to \psi_t \) uniformly on \([0, T]\) as \( k' \to 0 \). Thus, when \( h', k' \to 0 \), (4.16) yields

\[
- \int_0^T (u, \tilde{u} \psi_t)_H dt + \int_0^T \left\{ -\tilde{u}_0(\tilde{\zeta}, \tilde{\zeta}_x \psi)_H - \tilde{u}_0(\tilde{\zeta}, \tilde{\zeta}_x \psi)_H - (\tilde{\zeta}, \tilde{\eta}_x \psi)_H - (\eta, m \psi)_H + \tilde{f}(\tilde{\zeta}, \tilde{\eta}_x \psi)_H - (\tilde{\eta}_x + \tilde{\zeta}) \psi)_H + (\tilde{\eta}, \tilde{\zeta}_x \psi)_H \right\} dt \\
= (u_0, \tilde{u})_H \psi(0) + \int_0^T (f(t), \tilde{u}(t) \psi(t))_H dt.
\] (4.17)

Equation (4.17) is valid for all \( \tilde{u} \in D(A^*_T) \) and all \( \psi \in \mathcal{C}^1([0, T]) \) such that \( \psi(T) = 0 \). Reordering the terms in the second integral, we rewrite (4.17) as

\[
- \int_0^T (u, \tilde{u} \psi_t)_H dt + \int_0^T (u, A^*_T \tilde{u} \psi)_H dt = (u_0, \tilde{u})_H \psi(0) + \int_0^T (f(t), \tilde{u}(t) \psi(t))_H dt,
\] (4.18)

where \( A^*_T \tilde{u} = -(A_1 \tilde{u}, A_2 \tilde{u}, A_3 \tilde{u}) \) with

\[
\begin{align*}
A_1 \tilde{u} &= \tilde{u}_0 \tilde{\zeta}_x + \tilde{u}_0 \tilde{\eta}_x + \Phi_1 \tilde{\zeta}_x - \tilde{f}_\zeta, \\
A_2 \tilde{u} &= \tilde{u}_0 \tilde{\zeta}_x + \tilde{u}_0 \tilde{\eta}_x + \Phi_1 (\tilde{\zeta}_x - \tilde{\eta}_x) + \tilde{f}(\tilde{\zeta} + \tilde{\eta}), \\
A_3 \tilde{u} &= \tilde{u}_0 \tilde{\eta}_x + \tilde{u}_0 \tilde{\eta}_x - \Phi_1 \tilde{\eta}_x - \tilde{f}_\zeta.
\end{align*}
\] (4.19)

Choosing \( \psi \) compactly supported in \((0, T)\), we conclude that

\[
\frac{d}{dt}(u, \tilde{u})_H + (u, A^*_T \tilde{u})_H = (f, \tilde{u})_H,
\] (4.20)

in the distribution sense on \((0, T)\) for all \( \tilde{u} \in D(A^*_T) \). From (4.13), we see that \( du/dt \in L^2(0, T; H) \). Hence \( u \) is almost everywhere equal to a continuous function from \([0, T]\) to \( H \). Thus

\[
u \in C([0, T]; H).
\] (4.21)
and therefore $u(0)$ makes sense. Multiplying (4.20) by $\psi$, $\psi \in C^1([0, T])$, $\psi(T) = 0$, integrating over $(0, T)$, and then integrating by parts we find

$$-\int_0^T (u, \tilde{u})_H \psi_t dt + \int_0^T (u, A^*_S \tilde{u})_H \psi dt = (u(0), \tilde{u})_H \psi(0) + \int_0^T (f, \tilde{u})_H \psi dt,$$

(4.22)

$\forall \tilde{u} \in D(A^*_S)$. Comparing (4.18) and (4.22), we find that

$$(u_0, \tilde{u})_H \psi(0) = (u(0), \tilde{u})_H \psi(0), \quad \forall \tilde{u} \in D(A^*_S), \quad \forall \psi \in C^1([0, T]), \quad \psi(T) = 0.$$ 

Thus taking any $\psi$ such that $\psi(0) \neq 0$, we obtain

$$(u_0, \tilde{u})_H = (u(0), \tilde{u})_H, \quad \forall \tilde{u} \in D(A^*_S).$$

(4.23)

Because $D(A^*_S)$ is dense in $H$, we have $u(0) = u_0$. The proof of the theorem is hence complete.

**ACKNOWLEDGMENTS**

This work was supported in part by NSF grant DMS 0305110 and by the Research Fund of Indiana University. The authors are thankful to Professor Roger Temam for suggesting this problem as well as for his valuable comments and advice in solving it.

**REFERENCES**

13. R. Temam Well-posed initial value problems. Lectures given at the Summer School on Applications of Advanced Mathematical and Computational Methods to Atmospheric and Oceanic Problems (MCAO2003), National Center for Atmospheric Research (NCAR), Boulder, Colorado, USA, in [14].