SINGULAR PERTURBATION OF SEMI-LINEAR REACTION-CONVECTION EQUATIONS IN A CHANNEL AND NUMERICAL APPLICATIONS

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ABSTRACT. In this article, we investigate a way to analyze and approximate singularly perturbed convection-diffusion equations in a channel domain when a nonlinear reaction term with polynomial growth is present. We verify that the boundary layer structures are governed by certain simple recursive linear equations and this simplicity implies explicit pointwise and norm estimates. Furthermore, we can utilize the boundary layer structures (elements) in the finite elements discretizations which lead to the stability in the approximating systems and accurate approximation solutions with an economical mesh design, i.e., uniform mesh.

1. INTRODUCTION

Let $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ and $p: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that there exist constants $\gamma_l, \gamma_r, c_l, c_r > 0$, and $s \geq 1$ such that

$$\gamma_l |\xi|^{2s} - c_l \leq p(\xi) \xi \leq \gamma_r |\xi|^{2s} + c_r, \quad \forall \xi \in \mathbb{R}. \tag{1.1}$$

We are interested in improving the numerical solution of the following nonlinear singularly perturbed boundary value problems in a channel:

$$\begin{cases} -\epsilon \Delta u^\epsilon - \frac{\partial u^\epsilon}{\partial y} + p(u^\epsilon) = f & \text{in } \tilde{\Omega}, \\
 u^\epsilon = 0 & \text{at } y = 0, L_2, \\
 u^\epsilon \text{ is } L_1 - \text{periodic in } x, \end{cases} \tag{1.2}$$

where $0 < \epsilon \ll 1$, and $\tilde{\Omega} = \mathbb{R} \times (0, L_2)$. Without loss of generality we may assume that $p(0) = 0$; if not, we just subtract $p(0)$ from both sides of Eq. (1.2)$_1$. It is easily seen e.g. that the conditions (1.1) are satisfied when $p$ is a polynomial of degree $2s - 1$, $s$ integer $\geq 1$:

$$p(\xi) = a_{2s-1} \xi^{2s-1} + a_{2s-2} \xi^{2s-2} + \cdots + a_1 \xi, \quad a_{2s-1} > 0. \tag{1.3}$$

As $\epsilon$ becomes small, the solutions to such problems generally display near the boundaries thin transition layers called boundary layers, because the boundary conditions of the limit problem are different than those of the perturbed problem.
As $\epsilon \to 0$, the functions $u^\epsilon$ may display rapid variations near the boundary and their derivatives may become very large. Resolving boundary layers is very costly in numerical computations and simulations.

Indeed resolving boundary layers by the classical approximation methods requires very fine meshes near the boundaries, much smaller than the thickness of the boundary layers, which is very costly to realize in practice; see e.g., [E72], [ED66], [H95], [L73], [O91] and [VL57] for boundary layers and their asymptotic approximations, and see e.g. [RST96], [ST05] for numerical aspects of singularly perturbed problems.

Our aim in this article is to show how one can approximate the solutions of semi-linear reaction-convection problems by incorporating in the finite elements bases the ordinary boundary layer elements (BLE), an approach which has been developed in e.g. [CT02], [JT05a] for other types of problems. The BLEs enable us to use a uniform mesh for which we denote the mesh size by $h$ where $h = \max\{h_1, h_2\}$, $h_1 = 1/M$, $h_2 = 1/N$ being, respectively, the mesh size in the $x$-, and $y$- directions; note that we will not use mesh refinements near the boundary layers. Let us recall that the idea of incorporating the explicit forms of the singularities in the approximation space has been used in e.g. [CT02], [CTW00], [JT05a], [JT05b], [JT05c], [J05] and [J06] in the context of singular perturbations as in this article and in e.g. [CGS89], [HW97] in different contexts. In fact the idea of enriching the finite element space with elements containing the singularities was first introduced by [HK82], [HK83] although we were not aware of these articles when we started this series of articles. Note also that our earlier articles as well as this one contain many developments and ideas not pertaining to [HK82], [HK83]. Earlier than enriched subspaces in singular perturbation problems, exponentially fitted splines were introduced in [He77], which are adapted to the differential operators in one dimension, e.g. $-\epsilon d^2/\epsilon^2 - d/d\epsilon$; see also [OS91], [D99a], [D99b] for two dimensional extensions. The exponentially fitted splines absorb the singularities due to the small $\epsilon$ and these have similarities with the BLEs in the enriched finite element spaces. A more complete comparison of these articles with our own ones will appear in [JT05c].

We shall consider the Sobolev spaces $H^m(\Omega)$, $m$ integer, equipped with the semi-norm, $|u|_{H^m} = \left(\sum_{\alpha : |\alpha| = m} \int_\Omega |D^\alpha u|^2 dx dy\right)^{1/2}$, and the norm $\|u\|_{H^m} = \left(\sum_{\alpha : |\alpha| = m} |u|^2_{H^\alpha}\right)^{1/2}$. We define the corresponding inner product in the space $H^m(\Omega)$: $(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)$, where $(u, v) = \int_\Omega uv dx dy$.

Due to the nonlinearity, the problem (1.2) introduces new challenges when compared to the work that has been done for the linear problem (problem with the linear reaction term $p$) in e.g. [J06]. One of the most difficult factors that we must face in this case is that the solutions to the equations in the inner and outer expansions can not be achieved in explicit formulas because these equations are nonlinear. For example, in the case where $p(\xi) = \xi^3 + \xi^2 + \xi$, the limit problem of (1.2) when $\epsilon \to 0$ is the Abel ODE and it can not be solved explicitly in general. However, we are able to prove that, to some extent, all the results from the corresponding linear case, as in e.g. [J06], remain valid after being properly adapted and extended to our problem.

Our work is organized as follows. In Section 2 we start with the study of the boundary layer analysis using inner and outer asymptotic expansions and we derive the error estimates to all expansion orders. Next, in Section 3, we prove that the same boundary layer element as in the linear problem considered in [J06], when
added in the finite element space, absorbs the $H^2$- singularities of the solutions and gives a slightly different finite element error analysis as compared to the linear problem. Then, the pseudo-arclength continuation method is discussed in Section 4 to solve the finite element discretized nonlinear system. Lastly, the full details about the existence of smooth solutions for the limit problem of the model (1.2) when $\epsilon$ tends to 0 is given in Section 5.

2. Boundary Layer Analysis

We assume throughout this article that $f = f(x, y)$ is smooth and $L_1$ periodic in the $x$ direction; this means here that the periodic extension $\hat{f}$ of $f$ to $\hat{\Omega}$ is as smooth as needed. Beside the condition (1.1) for $p$, we assume furthermore that $p$ is strictly monotone increasing that is

$$
(p(\xi_1) - p(\xi_2))(\xi_1 - \xi_2) > 0, \forall \xi_1, \xi_2 \in \mathbb{R}, \xi_1 \neq \xi_2.
$$

We are looking for the solutions $u^\epsilon$ of (1.2) in the form $u^\epsilon = u^\epsilon_{\infty} + \theta^\epsilon_{\infty}$, where $u^\epsilon_{\infty}$, $\theta^\epsilon_{\infty}$ are respectively the outer and inner expansion series of $u^\epsilon$ that we now define and construct. We first start with the outer expansion:

2.1. Outer expansion (expansion outside the boundary layer). We consider the formal MacLaurin expansion of $p$

$$
p(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \frac{d^k p}{d\xi^k}(0),
$$
and the formal asymptotic expansion of $u^\epsilon$ with respect to $\epsilon$ which we denote by $u^\epsilon_{\infty}$

$$
u^\epsilon_{\infty} = \sum_{l=0}^{\infty} \epsilon^l u^l.
$$

By composition, we obtain the formal MacLaurin series expansion of $p(u^\epsilon_{\infty})$ with respect to $\epsilon$:

$$
p(u^\epsilon_{\infty}) = p \left( \sum_{j=0}^{\infty} \epsilon^j u^j \right) = \sum_{j=0}^{\infty} \epsilon^j r^j_u.
$$

It is not difficult to see that $r^j_u = r^j(u^j)$ where $u^j = (u^0, \ldots, u^j)$ for $j \geq 1$. Furthermore explicit forms of the $r^j_u$ are given by the Faà di Bruno formula, see e.g. [Com70] Chapter 3 Theorem C, that is:

$$
(2.2a) \quad r^j_u = \sum_{k=1}^{j} \frac{1}{j!} p^{(k)}(u^0) B_{j,k} \left( \left. \frac{d}{d\epsilon} u^\epsilon_{\infty} \right|_{\epsilon=0}, \ldots, \left. \frac{d^{(j-k+1)}}{d\epsilon^{(j-k+1)}} u^\epsilon_{\infty} \right|_{\epsilon=0} \right).
$$

where $p^{(k)}(u^0) = (d^k p/d\xi^k)(u^0)$ and the $B_{m,i}$ are the Bell polynomials given by

$$
(2.2b) \quad B_{j,k}(x_1, \ldots, x_{m-k+1}) = j! \sum_{(\alpha_1, \ldots, \alpha_{j-k+1})} \prod_{l=1}^{j-k+1} \frac{x_{l+1}^{\alpha_l}}{\alpha_l!(l!)^{\alpha_l}},
$$

with the $\alpha_l$ are all the nonnegative integers such that

$$
(2.2c) \quad \sum_{l=1}^{j-k+1} l\alpha_l = j; \sum_{l=1}^{j-k+1} \alpha_l = k.
$$
Thus the formulas for the $r^j_u$ can be deduced as
\begin{equation}
(2.2d) \quad r^j_u = \sum_{k=1}^{j} p^{(k)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_k)} \prod_{l=1}^{j-k+1} \frac{(u^l)^{\alpha_l}}{\alpha_l!}.
\end{equation}

The explicit formulas for $r^j_u$, $j = 0, 1, 2$ are e.g.
\begin{equation}
(2.3) \quad r^0_u = p(u^0), \quad r^1_u = p'(u^0)u^1, \quad r^2_u = p'(u^0)u^2 + p''(u^0)\frac{(u^1)^2}{2}.
\end{equation}

We obtain, by substituting the expansions of $u_{\infty}$ and $p(u_{\infty})$ into (1.2)1, the following equation
\begin{equation}
(2.4) \quad -\epsilon \sum_{j=0}^{\infty} \epsilon^j \Delta u^j - \sum_{j=0}^{\infty} \epsilon^j u^j_y + \sum_{j=0}^{\infty} \epsilon^j r^j_u = f.
\end{equation}

By a formal identification, we obtain the following equations for the $u^j, j \geq 0$
\begin{equation}
(2.5a) \quad \begin{cases} -u^0_y + r^0_u = f, \\ -u^j_y + r^j_u = \Delta u^j - 1, \quad j \geq 1. \end{cases}
\end{equation}

We choose to associate to these equations the boundary conditions
\begin{equation}
(2.5b) \quad u^j(x, L_2) = 0, \quad j \geq 0.
\end{equation}

Note that the choice of the boundary condition for $-d/dy$ will be eventually justified by the Convergence Theorem 2.1.

Existence and uniqueness of the $u^j$. We show in Section 5 that there exist unique functions $u^j$ solutions of (2.5) and that their periodic extensions to $\Omega$ are smooth. However, it is clear in general that the terms of the outer expansion do not satisfy the boundary condition (1.2)2 at $y = 0$. To address these discrepancies we now introduce the ordinary boundary layers given by the inner expansion below.

2.2. Inner expansion (expansion inside the boundary layer). We consider the formal asymptotic expansion $\sum_{j=0}^{\infty} \epsilon^j \theta^j$ of $u'$ in the stretched variable $y'/\epsilon$ which we denote by $\theta_{\infty} = \sum_{j=0}^{\infty} \epsilon^j \theta^j$, where $\theta^j$ is called the corrector at order $j$. Here $\theta_{\infty}$ is set to guarantee that $u_{\infty} + \theta_{\infty}$ is formally a solution of (1.2), see [O74].

The equation for $\theta_{\infty}$ is hence obtained as
\begin{equation}
(2.6) \quad \begin{cases} -\epsilon \Delta \theta_{\infty} - \frac{\partial \theta_{\infty}}{\partial y} + p(u_{\infty} + \theta_{\infty}) - p(u_{\infty}) = 0, \\ \theta_{\infty}(\cdot, 0) = -u_{\infty}(\cdot, 0), \quad \theta_{\infty}(\cdot, L_2) = -u_{\infty}(\cdot, L_2) = 0. \end{cases}
\end{equation}

Let us denote by $r^j_{u+\theta}$ the $j$-th coefficient in the MacLaurin expansion with respect to $\epsilon$ of $p(u_{\infty} + \theta_{\infty})$; as before these coefficients are found by the Faà di Bruno formula and they read
\begin{equation}
(2.7) \quad r^j_{u+\theta} = \sum_{k=1}^{j} p^{(k)}(u^0 + \theta^0) \sum_{(\alpha_1, \ldots, \alpha_j)} \prod_{l=1}^{j-k+1} \frac{(u^l + \theta^l)^{\alpha_l}}{\alpha_l!}.
\end{equation}
since they are solutions to the boundary-valued linear ODEs of the form

\begin{equation}
\begin{aligned}
    r^0_{u+\theta} &= p(u^0 + \theta^0), \\
    r^1_{u+\theta} &= p'(u^0 + \theta^1)(u^1 + \theta^1), \\
    r^2_{u+\theta} &= p'(u^0 + \theta^0)(u^2 + \theta^2) + p''(u^0 + \theta^0)(u^1 + \theta^1)^2.
\end{aligned}
\end{equation}

We thus have from (2.6) that

\begin{equation}
\begin{aligned}
    -\epsilon \sum_{j=0}^{\infty} \epsilon^j \Delta \theta^j - \sum_{j=0}^{\infty} \epsilon^j \frac{\partial \theta^j}{\partial y} + \sum_{j=0}^{\infty} \epsilon^j (r^i_{u+\theta} - r^i_u) = 0, \\
    \sum_{j=0}^{\infty} \epsilon^j \theta^j(\cdot,0) = -\sum_{j=0}^{\infty} \epsilon^j u^j(\cdot,0), \\
    \sum_{j=0}^{\infty} \epsilon^j \theta^j(\cdot, L_2) = 0.
\end{aligned}
\end{equation}

Setting \( \bar{y} = y/\epsilon \), we find by a formal identification of the powers of \( \epsilon \) that

\begin{equation}
\begin{aligned}
    -\frac{\partial^2 \theta^0}{\partial \bar{y}^2} - \frac{\partial \theta^0}{\partial \bar{y}} = 0, \\
    -\frac{\partial^2 \theta^1}{\partial \bar{y}^2} - \frac{\partial \theta^1}{\partial \bar{y}} + r^0_{u+\theta} - r^0_u = 0, \\
    -\frac{\partial^2 \theta^j}{\partial \bar{y}^2} - \frac{\partial \theta^j}{\partial \bar{y}} + r^{j-1}_{u+\theta} - r^{j-1}_u - \frac{\partial^2 \theta^{j-2}}{\partial \bar{x}^2} = 0, \forall j \geq 2, \\
    \theta^j(\cdot,0) = -u^j(\cdot,0), \theta^j(\cdot, L_2/\epsilon) = 0, \forall j \geq 0.
\end{aligned}
\end{equation}

It is also understood in (2.10) that all functions are periodic in \( x \) with period \( L_1 \).

The equations for the correctors are also seen in the variable \( y \) as

\begin{equation}
\begin{aligned}
    -\epsilon \frac{\partial^2 \theta^j}{\partial \bar{y}^2} - \frac{\partial \theta^j}{\partial \bar{y}} + \frac{1}{\epsilon} \left( r^{j-1}_{u+\theta} - r^{j-1}_u - \frac{\partial^2 \theta^{j-2}}{\partial \bar{x}^2} \right) = 0, \quad j \geq 0.
\end{aligned}
\end{equation}

Here we adopt the notation convention \( r^j_u = r^j_{u+\theta} = \partial^2 \theta^j / \partial x^2 = 0 \) for \( j \leq -1 \). As in the linear case, see e.g. [J06], where the terms \( r^{j-1}_u, r^{j-1}_{u+\theta} \) are not present in the above equations for the correctors, these correctors are exponentially decaying in \( y \), see Lemma 2.1 below. To prove this lemma, we introduce the simplified functions \( \overline{\theta}^j \) that are solutions of (2.10) but with the boundary conditions \( \lim_{\bar{y} \to \infty} \overline{\theta}^j(\cdot, \bar{y}) = 0 \)

\begin{equation}
\begin{aligned}
    \theta^0(x,y) = -u^0(x,0)e^{-y/\epsilon} - e^{-L_2/\epsilon} = -u^0(x,0)e^{-y/\epsilon} + \text{e.s.t.} = \overline{\theta}^0 + \text{e.s.t.}
\end{aligned}
\end{equation}

Knowing from the outer expansion that the \( u^j, j \geq 0 \) are smooth and \( L_1 \) periodic in \( x \), we quickly observe that \( \theta^0 \) is also smooth and periodic in \( x \) direction with period \( L_1 \). By induction, we then see that the \( \theta^j \) are all smooth and \( L_1 \) periodic in \( x \) since they are solutions to the boundary-valued linear ODEs of the form

\begin{equation}
\begin{aligned}
    \frac{\partial^2 \theta}{\partial \bar{y}^2}(x,y) + \frac{\partial \theta}{\partial \bar{y}}(x,y) = g(x,y), \\
    \theta(x,0) = a(x), \theta(x,L_2) = b(x),
\end{aligned}
\end{equation}

where the \( \alpha_l \) are defined in (2.2c). For instance, the formulas for \( r^0_{u+\theta}, r^1_{u+\theta}, r^2_{u+\theta} \) are easily found as

\begin{equation}
\begin{aligned}
    r^0_{u+\theta} &= p(u^0 + \theta^0), \\
    r^1_{u+\theta} &= p'(u^0 + \theta^1)(u^1 + \theta^1), \\
    r^2_{u+\theta} &= p'(u^0 + \theta^0)(u^2 + \theta^2) + p''(u^0 + \theta^0)(u^1 + \theta^1)^2.
\end{aligned}
\end{equation}
where \( g(x, y), a(x), b(x) \) are all smooth and \( L_1 \) periodic in \( x \). Furthermore, these correctors \( \theta^j \) are also exponentially decaying in \( y \) as shown in the following lemma:

**Lemma 2.1.** The solutions \( \theta^j \) to (2.10), \( \forall j \geq 0 \), satisfy, for all \( c, 0 < c < 1 \)

\[
\left| \frac{\partial^{m+l}\theta^j}{\partial x^m\partial y^l}(\cdot, y/\epsilon) \right| \leq \kappa_{j,l,m} e^{-c \epsilon/y}, \quad \text{for all } l, m \geq 0,
\]

where the \( \kappa_{j,l,m} \) are constants independent of \( \epsilon \) but dependent on \( c \).

**Proof.** We will prove (2.13) for the approximated solutions \( \tilde{\theta}^j \) and we only prove it in detail for the case \( l = 0 \), since for the general case where \( l \geq 1 \) we can obtain the result in a similar way by differentiating (2.19) below. For the case \( l = 0 \), (2.13) for the \( \tilde{\theta}^j \) can be restated as

\[
\left| \frac{\partial^m}{\partial x^m} \tilde{\theta}^j(\cdot, y/\epsilon) \right| \leq \kappa_{jm} e^{-c \epsilon/y}, \quad \text{for all } m \geq 0,
\]

where \( \kappa_{jm} = \kappa_{j,m,0} \) is a generic constant independent of \( \epsilon \) but dependent on \( c, 0 < c < 1 \). To establish (2.14), we will prove by induction that

\[
\left| \frac{\partial^m}{\partial x^m} \tilde{\theta}^i(\cdot, \bar{y}) \right| \leq \kappa_{j,m} P_j(\bar{y}) e^{-\bar{y}}, \quad \text{for all } m \geq 0,
\]

where \( P_j(\bar{y}) \) is a polynomial of order \( j \) in the stretched variable \( \bar{y} = y/\epsilon \) which might be different at each instance in this proof. It is obvious that \( \bar{\theta}^1 \) satisfies (2.15) with \( \kappa_{0m} = \sup_{\bar{y}} |\partial^m / \partial x^m u^0(x,0)| \). For \( j = 1 \), by direct integration we find the solution \( \tilde{\theta}^1 \), which is the sum of the general solution \( \tilde{\theta}^1_h \) to the corresponding homogenous equation and a particular solution \( \tilde{\theta}^1_p \) of the whole equation:

\[
\tilde{\theta}^1 = \tilde{\theta}^1_h + \tilde{\theta}^1_p = C_1(x) e^{-\bar{y}} - \int_{\bar{y}}^\infty \int_0^t e^{-(t-s)} (r_{u+\theta}^0 - r_u^0) \, ds \, dt,
\]

where \( C_1(x) = -u^1(x,0) - \tilde{\theta}^1_h(x,0) \) so that \( \tilde{\theta}^1(x,0) = -u^1(x,0) \). Thus

\[
\frac{\partial^m \tilde{\theta}^1}{\partial x^m} = C_1^{(m)}(x) e^{-\bar{y}} - \int_{\bar{y}}^\infty \int_0^t e^{-(t-s)} \frac{\partial^m}{\partial x^m} (r_{u+\theta}^0 - r_u^0) \, ds \, dt.
\]

Using the Faà di Bruno formula, we write the term \( \partial^m (r_{u+\theta}^0 - r_u^0) / \partial x^m \) as follows:

\[
\frac{\partial^m}{\partial x^m} (r_{u+\theta}^0 - r_u^0) = \frac{\partial^m}{\partial x^m} (p(u^0 + \theta^0) - p(u^0))
\]

\[
= \sum_{i=1}^m p^{(i)}(u^0 + \theta^0) B_{m,i} \left( \frac{\partial}{\partial x}(u^0 + \theta^0), \ldots, \frac{\partial^{m+i-1}}{\partial x^{m+i-1}}(u^0 + \theta^0) \right)
\]

\[
- \sum_{i=1}^m p^{(i)}(u^0) B_{m,i} \left( \frac{\partial}{\partial x} u^0, \ldots, \frac{\partial^{m+i-1}}{\partial x^{m+i-1}} u^0 \right)
\]

\[
= \sum_{i=1}^m (I_{mi}^1 + I_{mi}^2),
\]

\(^{1}\text{A functions } g' \text{ is said to be an exponentially small term, denoted e.s.t., if there exists } \alpha \in (0,1) \text{ and } \alpha' > 0 \text{ such that for every } k \geq 0, \text{ there exists a constant } c_{\alpha,\alpha',k} > 0 \text{ with } \|g'\|_{H^k} \leq c_{\alpha,\alpha',k} e^{-\alpha' k}; e^{-(1+\varepsilon)/\epsilon} \text{ is an example of e.s.t.}\)
where the $B_{m,i}$ are as in (2.2b) and the $I_{mi}', I_{mi}''$ are given by

$$I_{mi}' = \left( p^{(i)}(u^0 + \theta^0) - p^{(i)}(u^0) \right) B_{m,i} \left( \frac{\partial}{\partial x}(u^0 + \theta^0), \ldots, \frac{\partial^{m-i+1}}{\partial x^{m-i+1}}(u^0 + \theta^0) \right),$$

$$I_{mi}'' = p^{(i)}(u^0).$$

We now estimate $I_{mi}'', I_{mi}'''$. For $I_{mi}'$, we use the mean value theorem, and we find $\rho^i = \rho^i(x, y)$, $0 < \rho^i < 1$ such that

$$I_{mi}' = p^{(i+1)}(u^0 + \rho^i\theta^0) B_{m,i} \left( \frac{\partial}{\partial x}(u^0 + \theta^0), \ldots, \frac{\partial^{m-i+1}}{\partial x^{m-i+1}}(u^0 + \theta^0) \right) \theta^0.$$

Since the functions $\frac{\partial}{\partial x}u^0, \frac{\partial}{\partial x}\theta^0$, $0 \leq l \leq m - i + 1$, are uniformly bounded on $\Omega$, $\theta^0$ is exponentially decaying in $y$ like $e^{-y/\epsilon}$ and $p^{(i+1)}$ is continuous, there exists a $\kappa_{1m}$ independent of $\epsilon$ such that $|I_{mi}'| \leq \kappa_{1m} e^{-y}$.

To estimate the term $I_{mi}'''$, using the binomial Theorem and then expanding all the product terms in the polynomial, we see that

$$I_{mi}''' = p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} \frac{(\frac{\partial}{\partial x}^l u^0 + \theta^0)}{\alpha_l! (l)! \alpha_l} - \prod_{l=1}^{m-i+1} \frac{(\frac{\partial}{\partial x}^l u^0)}{\alpha_l! (l)! \alpha_l} \right)$$

$$= p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} \frac{(\frac{\partial}{\partial x}^l u^0 + \theta^0)}{\alpha_l! (l)! \alpha_l} - \prod_{l=1}^{m-i+1} \frac{(\frac{\partial}{\partial x}^l u^0)}{\alpha_l! (l)! \alpha_l} \right)$$

$$= p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} (\alpha_l! (l) \alpha_l)^{-1} \sum_{j=0}^{\alpha_l-1} \left( \frac{\partial}{\partial x}^j u^0 \right) (\frac{\partial}{\partial x}^j \theta^0) \alpha_l^{-j} \right)$$

$$= p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} \frac{(\partial u^0)}{\partial x^l} \right) \left( \prod_{l=1}^{m-i+1} \frac{(\partial \theta^0)}{\partial x^l} \right) \alpha_l^{-j} \left( \frac{\partial}{\partial x^j} u^0 \right) \left( \frac{\partial}{\partial x^j} \theta^0 \right)$$

$$= p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} \frac{(\partial u^0)}{\partial x^l} \right) \left( \prod_{l=1}^{m-i+1} \frac{(\partial \theta^0)}{\partial x^l} \right) \alpha_l^{-j} \left( \frac{\partial}{\partial x^j} u^0 \right) \left( \frac{\partial}{\partial x^j} \theta^0 \right)$$

$$= p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} \frac{(\partial u^0)}{\partial x^l} \right) \left( \prod_{l=1}^{m-i+1} \frac{(\partial \theta^0)}{\partial x^l} \right) \alpha_l^{-j} \left( \frac{\partial}{\partial x^j} u^0 \right) \left( \frac{\partial}{\partial x^j} \theta^0 \right)$$

$$= p^{(i)}(u^0) \sum_{(\alpha_1, \ldots, \alpha_{m-i+1})} \left( \prod_{l=1}^{m-i+1} \frac{(\partial u^0)}{\partial x^l} \right) \left( \prod_{l=1}^{m-i+1} \frac{(\partial \theta^0)}{\partial x^l} \right) \alpha_l^{-j} \left( \frac{\partial}{\partial x^j} u^0 \right) \left( \frac{\partial}{\partial x^j} \theta^0 \right)$$

We know that the $\frac{\partial}{\partial x}u^0, \frac{\partial}{\partial x}\theta^0$, $0 \leq i \leq m$, are uniformly bounded on $\Omega$, then so are the $F_{mii}(u^0, \theta^0)$. Furthermore, the $\frac{\partial}{\partial x}u^0, \frac{\partial}{\partial x}\theta^0$ are exponentially decaying in $y$ like $e^{-y/\epsilon}$ and we thus find that $|I_{mi}'''| \leq \kappa_{0m} e^{-y}$. We hence deduce from (2.17) that

$$\left| \frac{\partial^m}{\partial x^m} \tilde{y}(x, \tilde{y}) \right| \leq \left| C_1(x) e^{-\tilde{y}} \right| + \int_{\tilde{y}}^\infty \int_0^t e^{-(t-s)} \kappa_{0m} e^s ds dt$$

$$\leq \kappa_{1m} e^{-\tilde{y}} + \kappa_{1m} \int_{\tilde{y}}^\infty e^{-t} dt \leq \kappa_{1m}(\tilde{y} + 1) e^{-\tilde{y}} = \kappa_{1m} F_1(\tilde{y}) e^{-\tilde{y}}.$$
homogeneous equation and a particular solution \( \tilde{\theta}_j^{i+1} \) of the full equation:

\[(2.18)\]
\[\tilde{\theta}_j^{i+1} = \tilde{\theta}_h^{i+1} + \tilde{\theta}_p^{i+1} = C_{j+1}(x)e^{-\bar{y}} - \int_0^\infty t \int_0^t e^{-(t-s)} \left( r_u^j \frac{\partial^2 \theta^{i-1}}{\partial x^2} \right) ds dt,
\]

where \( C_{j+1}(x) = -u_j^{i+1}(x,0) - \tilde{\theta}_p^{i+1}(x,0) \). The partial derivative of order \( m \) in \( x \) is easily seen as

\[(2.19)\]
\[\frac{\partial^m}{\partial x^m}(r_u^j - r_u^j) = \frac{\partial^m}{\partial x^m}C_{j+1}(x)e^{-\bar{y}} - \int_0^\infty t \int_0^t e^{-(t-s)} \left( \frac{\partial^m}{\partial x^m}(r_u^j - r_u^j) + \frac{\partial^m+2\theta^{i-1}}{\partial x^{m+2}} \right) ds dt,
\]

Like in the case \( j = 1 \), we mainly need to prove that the partial derivatives in \( x \) of the particular solution are exponentially decaying in \( y \). We first estimate the term \( \frac{\partial^m}{\partial x^m}(r_u^j - r_u^j)/\partial x^m \) in (2.18); to this end, we infer from (2.2d) and (2.7) that

\[
\frac{\partial^m}{\partial x^m}(r_u^j - r_u^j)
= \frac{\partial^m}{\partial x^m} \sum_{k=1}^j \sum_{l=1}^{\alpha_1 \ldots \alpha_{j+1}} \left( p^{(k)}(u^0 + \theta^0) \prod_{l=1}^{j+1} \frac{(u^l + \theta^l)^{\alpha_l}}{\alpha_l!} - p^{(k)}(u^0) \prod_{l=1}^{j+1} \frac{(u^l)^{\alpha_l}}{\alpha_l!} \right)
= \sum_{k=1}^j \sum_{l=1}^{\alpha_1 \ldots \alpha_{j+1}} \left( \frac{\partial^m}{\partial x^m}J_{jk}^1 + \frac{\partial^m}{\partial x^m}J_{jk}^2 \right),
\]

where

\[
J_{jk}^1 = \left( p^{(k)}(u^0 + \theta^0) - p^{(k)}(u^0) \right) \prod_{l=1}^{j+1} \frac{(u^l + \theta^l)^{\alpha_l}}{\alpha_l!},
\]

\[
J_{jk}^2 = p^{(k)}(u^0) \prod_{l=1}^{j+1} \frac{(u^l + \theta^l)^{\alpha_l}}{\alpha_l!} - \prod_{l=1}^{j+1} \frac{(u^l)^{\alpha_l}}{\alpha_l!}.
\]

We now estimate \( \frac{\partial^m}{\partial x^m}J_{jk}^1 \)/\( \partial x^m \), \( \frac{\partial^m}{\partial x^m}J_{jk}^2 \)/\( \partial x^m \). For \( \frac{\partial^m}{\partial x^m}J_{jk}^1 \)/\( \partial x^m \), using the mean value theorem, we find \( \rho^k = \rho^k(x, y), \ 0 < \rho^k < 1 \) such that

\[
\frac{\partial^m}{\partial x^m}J_{jk}^1 = \frac{\partial^m}{\partial x^m} \left[ p^{(k+1)}(\rho^k u^0 + \theta^0) \prod_{l=1}^{j+1} \frac{(u^l + \theta^l)^{\alpha_l}}{\alpha_l!} u^0 \right]
= \sum_{l=0}^m \left( \sum_{l=0}^m \frac{\partial^l}{\partial x^l}G_{jkl}(u^0, u^l, \theta^0, \theta^l) \frac{\partial^m-l}{\partial x^{m-l}} u^0, \right)
\]

where

\[
G_{jkl}(u^0, u^l, \theta^0, \theta^l) = p^{(k+1)}(\rho^k u^0 + \theta^0) \prod_{l=1}^{j+1} \frac{(u^l + \theta^l)^{\alpha_l}}{\alpha_l!}.
\]

It is easily seen by the induction hypotheses that the partial derivatives of \( G_{jkl}(u^0, u^l, \theta^0, \theta^l) \), \( \partial^l G_{jkl}(u^0, u^l, \theta^0, \theta^l)/\partial x^l \) for \( 0 \leq l \leq m \), are uniformly bounded on \( \Omega \). Since the \( \partial^l u^0/\partial x^l \), \( 0 \leq m \), are exponentially decaying in \( y \), we can find a constant \( \kappa_{mj} \) and a polynomial \( P_j(\bar{y}) \) such that \( |\partial^m J_{jk}^1/\partial x^m| \leq \kappa_{mj} P_j(\bar{y}) e^{-\bar{y}} \).

To estimate \( \frac{\partial^m}{\partial x^m}J_{jk}^2 \)/\( \partial x^m \), likewise the estimates for the \( J_{m+1}^2 \), we use the binomial
Theorem, then expand all the product terms and obtain
\[ J_{jk}^2 = p^{(k)}(u^0) \prod_{l=1}^{j-k+1} (a_l)^{-1} \sum_{j_l=1}^{a_j-1} (a_l)^{j_l} (\theta^l)_{a_l-j_l} = H_{jkl}(u_l, \theta_l)\theta^l, \]
where \( H_{jkl}(u_l, \theta_l) = p^{(k)}(u^0) \prod_{l=1}^{j-k+1} (a_l)^{-1} \sum_{j_l=1}^{a_j-1} (a_l)^{j_l} (\theta^l)_{a_l-j_l-1}. \) Therefore
\[ \frac{\partial^m}{\partial x^m} J_{jk}^2 = \sum_{i=0}^{m} \binom{m}{i} H_{jkl}(u_l, \theta_l) \frac{\partial^{m-i}}{\partial x^{m-i}} \theta^l. \]
By the induction hypotheses, the \( \partial^l H_{jkl}(u_l, \theta_l)/\partial x^l \) are uniformly bounded and the \( \partial^{m-i} \theta^l/\partial x^{m-i} \) are exponentially decaying in \( y \in \Omega \) for all \( 0 \leq i \leq m \). We then deduce that
\[ |\partial^m J_{jk}^2/\partial x^m| \leq \kappa_{jm} P_j(\bar{y}) e^{-\bar{y}} \text{ for some } \kappa_{jm} > 0. \] Therefore we have
\[ \left| \frac{\partial^m}{\partial x^m} (r_j^{i+\theta} - r_j^u) \right| \leq \sum_{k=1}^{j} \sum_{l=1}^{a_j} \kappa_{jm} P_j(\bar{y}) e^{-\bar{y}} \leq \kappa_{j+1} P_j(\bar{y}) e^{-\bar{y}}. \]
It is easy to see by induction that
\[ |\partial^{m+2} \theta^{j-1}/\partial x^{m+2}| \leq \kappa_{j-1} P_{j-1}(\bar{y}) e^{-\bar{y}}. \] We thus obtain from (2.19) that
\[ \left| \frac{\partial^m}{\partial x^m} \bar{H}_{jkl}(\cdot, \bar{y}) \right| \leq \kappa_{j+1} \int_{\bar{y}}^{\infty} e^{-t} \int_0^t e^s (P_j(s) + P_{j-1}(s)) e^{-s} ds dt \]
\[ = \kappa_{j+1} \int_{\bar{y}}^{\infty} e^{-t} \int_0^t (P_j(s) + P_{j-1}(s)) ds dt \]
\[ = \kappa_{j+1} \int_{\bar{y}}^{\infty} e^{-t} P_{j+1}(t) dt = \kappa_{j+1} P_{j+1}(\bar{y}) e^{-\bar{y}}. \]
The proof of Lemma 2.1 hence is complete. \( \square \)

Before proving the main results, we need one more estimate

**Lemma 2.2.** For every \( n \geq 0 \), there exists a constant \( \kappa = \kappa_n > 0 \) independent of \( \epsilon \) such that
\[ \left| p(u_{en} + \theta_{en}) - \sum_{j=0}^{n} \epsilon^j r_j^{i+\theta} \right| \leq \kappa \epsilon^{n+1}, \quad (L^2(\Omega)) \]
where the \( r_j^{i+\theta} \) are defined at (2.7) and the \( u_{en}, \theta_{en} \) are defined by:
\[ u_{en} = \sum_{j=0}^{n} \epsilon^j u_j, \quad \theta_{en} = \sum_{j=0}^{n} \epsilon^j \theta_j. \]

**Proof.** Applying the MacLaurin expansion with respect to \( \epsilon \) for the function \( q(\epsilon) := p(u_{en} + \theta_{en}) = p(\sum_{j=0}^{n} \epsilon^j (u_j + \theta_j)) \), we find
\[ q(\epsilon) = \sum_{j=0}^{n} \epsilon^j r_j^{i+\theta} + R_n(\epsilon). \]
Here $R_n(\epsilon) = \epsilon^{n+1}q(n+1)(\epsilon^*)/(n+1)!$ for $0 < \epsilon^* < \epsilon$, and the $r^j_{u_{en} + \theta_{en}} = q^{(j)}(0)/j!$, 0 $\leq j \leq n$ are found by the Faà di Bruno formula as follows:

$$r^j_{u_{en} + \theta_{en}} = \sum_{k=1}^{j} p^{(k)}(u^0 + \theta^0) B_{j,k} \left( \frac{d}{d\epsilon} (u_{en} + \theta_{en})|_{\epsilon=0}, \cdots, \frac{d^{j-k+1}}{d\epsilon^{j-k+1}} (u_{en} + \theta_{en})|_{\epsilon=0} \right),$$

where the $B_{j,k}$ are as in (2.2b). We easily see from the above formula that $r^j_{u_{en} + \theta_{en}} = r^j_{u + \theta}$ for $0 \leq j \leq n$, and therefore

$$q(\epsilon) = \sum_{j=0}^{n} \epsilon^j r^j_{u + \theta} + R_n(\epsilon).$$

Hence

$$p(u_{en} + \theta_{en}) - \sum_{j=0}^{n} \epsilon^j r^j_{u + \theta} = R_n(\epsilon).$$

It is left to prove that $|R_n(\epsilon)|_{L^2(\Omega)} \leq \kappa \epsilon^{n+1}$. We find again by the Faà di Bruno formula that

$$R_n(\epsilon) = \epsilon^{n+1} \sum_{k=1}^{n+1} \sum_{(\alpha_1, \ldots, \alpha_{n-k+2}) \text{ such that } \alpha_1 + 2\alpha_2 + \cdots + (n-k+2)\alpha_{n-k+2} = n+1} \frac{p^{(k)}}{\alpha_1!\alpha_2!\cdots\alpha_{n-k+2}!} \left( \frac{\sum_{j=1}^{n} \epsilon^{\alpha_1} \cdots \epsilon^{\alpha_{n-k+2}} (u^j + \theta^j) \cdots (u^j + \theta^j)}{\alpha_1!\alpha_2!\cdots\alpha_{n-k+2}!} \right),$$

where $(\alpha_1, \ldots, \alpha_{n-k+2})$ are nonnegative integers such that $\alpha_1 + 2\alpha_2 + \cdots + (n-k+2)\alpha_{n-k+2} = n+1$, $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-k+2} = k$ and $\alpha_l \leq n$, for $1 \leq l \leq n-k+2$. Since the $u^j, \theta^j, 0 \leq j \leq n$ are uniformly bounded functions on $\Omega$ and the $p^{(k)}$, $1 \leq k \leq n+1$ are continuous from $\mathbb{R}$ to $\mathbb{R}$, we can find $\kappa = \kappa_n$ independent of $\epsilon$ such that $|R_n(\epsilon)|_{L^2(\Omega)} \leq \kappa \epsilon^{n+1}$. Hence (2.21) follows. \hfill \square

We are now ready to state and prove the following main results

2.3. Convergence and error estimates. For every $n \geq 0$, we define the following asymptotic error at order $n$:

$$w_{en} = u^\epsilon - u_{en} - \theta_{en},$$

where the $u_{en}, \theta_{en}$ are as in (2.22) and $u^\epsilon$ is the solution of (1.2) in $\Omega$. The following error estimates are valid for all orders of the expansion:

**Theorem 2.1.** Let the $w_{en}$ be as in (2.23). For every $n \geq 0$, we have

$$\begin{align*}
|w_{en}|_{L^2(\Omega)} &\leq \kappa \epsilon^{n+1/2}, \\
|w_{en}|_{H^1(\Omega)} &\leq \kappa \epsilon^n, \\
|w_{en}|_{H^2(\Omega)} &\leq \kappa \epsilon^{n-1},
\end{align*}$$

where $\kappa$ is a constant dependent on $n$ but independent of $\epsilon$.\hfill \square
We multiply (2.27) by \( e \) and consider boundary conditions for \( w \):

\[
-\epsilon \Delta u_{en} - \frac{\partial u_{en}}{\partial y} + \sum_{j=0}^{n} e^{j} r_{u}^{j} = f - e^{n+1} \Delta u^{n},
\]

\[
-\epsilon \Delta \theta_{en} - \frac{\partial \theta_{en}}{\partial y} + p(\theta_{en} + u_{en}) - \sum_{j=0}^{n} e^{j} r_{u}^{j} = R_{n},
\]

where

\[
R_{n} = -e^{n+1} \frac{\partial^{2} \theta}{\partial x^{2}} - \frac{\partial^{2} \theta_{n-1}}{\partial x^{2}} - e^{n} p(u_{en} + \theta_{en}) - \sum_{j=0}^{n-1} e^{j} r_{u+\theta}^{j} - e^{n} \left( r_{u+\theta}^{n} - r_{u}^{n} - \frac{\partial^{2} \theta_{n-1}}{\partial x^{2}} \right).
\]

Performing (1.2) \( - (2.26) - (2.25) \), we obtain

\[
-\epsilon \Delta w_{en} - \frac{\partial w_{en}}{\partial y} + p(u') - p(\theta_{en} + u_{en}) = R_{n}^{2} := e^{n+1} \Delta u^{n} + R_{n}.
\]

Since the \( u^{j}, \theta^{j}, j \geq 0 \) are \( L_{1} \) periodic in \( x \), from (1.2) \( 2,3 \), (2.5) \( 1 \) and (2.9), the boundary conditions for \( w_{en} \) are found to be

\[
\begin{aligned}
\begin{cases}
 w_{en}(x,0) = w_{en}(x,L_{2}) = 0, \\
w_{en} \text{ is } L_{1} \text{ periodic in } x.
\end{cases}
\end{aligned}
\]

We multiply (2.27) by \( e^{y} w_{en} \) and integrate over \( \Omega \); also noticing that since \( w_{en} \) is periodic in \( x \) and zero at \( y = 0, L_{2} \), we have by Green’s formula that

\[
-\epsilon \int_{\Omega} \Delta w_{en} e^{y} w_{en} = \epsilon \int_{\Omega} e^{y} |\nabla w_{en}|^{2} + \frac{\epsilon}{2} \int_{\Omega} e^{y} \frac{\partial w_{en}}{\partial y} = \epsilon \int_{\Omega} e^{y} |\nabla w_{en}|^{2} - \frac{\epsilon}{2} \int_{\Omega} e^{y} w_{en}^{2},
\]

and also

\[
- \int_{\Omega} \frac{\partial w_{en}}{\partial y} e^{y} w_{en} = - \frac{1}{2} \int_{\Omega} e^{y} \frac{\partial w_{en}}{\partial y} = \frac{1}{2} \int_{\Omega} e^{y} w_{en}^{2}.
\]

Hence we have

\[
\epsilon \int_{\Omega} e^{y} |\nabla w_{en}|^{2} + \left( \frac{1 - \epsilon}{2} \right) \int_{\Omega} e^{y} w_{en}^{2} + \int_{\Omega} \left( p(u') - p(\theta_{en} + u_{en}) \right) e^{y} w_{en} = \int_{\Omega} R_{n}^{2} e^{y} w_{en}.
\]

By (2.1) \( 1 \) and Taylor’s formula at order 0, the third term in the left-hand side of

(2.29) is nonnegative. Furthermore, the term on the right-hand side can be bounded by the Cauchy-Schwarz inequality as follows

\[
\int_{\Omega} R_{n}^{2} e^{y} w_{en} \leq \frac{1}{4} \int_{\Omega} e^{y} |w_{en}|^{2} + \int_{\Omega} e^{y} |R_{n}^{2}|^{2}.
\]

Hence we obtain

\[
\epsilon \int_{\Omega} e^{y} |\nabla w_{en}|^{2} + \left( \frac{1 - \epsilon}{2} \right) \int_{\Omega} e^{y} |w_{en}|^{2} \leq \int_{\Omega} e^{y} |R_{n}^{2}|^{2}.
\]

To bound \( R_{n}^{2} \) in the norm of \( L^{2}(\Omega) \), from Lemma 2.2 we firstly notice that \(|p(u_{en} + \theta_{en}) - \sum_{j=0}^{n} e^{j} r_{u+\theta}^{j} \|L^{2}(\Omega) \leq \kappa e^{n+1}|\). Besides, using the property that \(|y^{n} e^{-cy/\epsilon^{2}} \|L^{2}(0,\alpha) \leq \kappa_{n,\alpha} e^{\epsilon+1/2}|\) and using (2.20), we find \(|r_{u+\theta}^{n} - r_{u}^{n}|_{L^{2}(\Omega)} \leq \kappa_{1/2}|\) and therefore

\[
|R_{n}^{2}|_{L^{2}(\Omega)} \leq \kappa e^{n+1/2}.
\]
Hence, we obtain from (2.30)
\[
\epsilon\|w_{en}\|_{H^1(\Omega)}^2 + |w_{en}|_{L^2(\Omega)}^2 \leq \kappa \epsilon^{2n+1},
\]
where \(\kappa\) depends on \(u^n, \theta^n\) but is independent of \(\epsilon\). The theorem follows. \(\square\)

3. Finite Elements Discretizations

In this section, we justify the utilization in the finite elements discretizations of the so-called boundary layer elements (BLE) \(\psi_0^*, \psi_0\) (see Figure 1 below) which are
\[
\psi_0^* = -\exp(-y/\epsilon) - (1 - \exp(-1/\epsilon))y + 1,
\]
(3.2)
\[
\psi_0 = (-\exp(-y/\epsilon) - (1 - \exp(-h_2/\epsilon))y/h_2 + 1)\chi_{(0,h_2)}(y),
\]
where \(\chi_{[\alpha, \beta]}\) is the characteristic function of \([\alpha, \beta]\); both functions, \(\psi_0^*, \psi_0\), belong to \(H^1_0(0,L_2)\) but \(\psi_0\) has a small compact support.

Setting \(\mathcal{V} = \{v \in H^1(\Omega) \mid v(0, \cdot) = v(L_1, \cdot), \ v(\cdot, 0) = v(\cdot, L_2) = 0\}\), we will take into account the weak formulation of our model problem (1.2). Notice that the space \(\mathcal{V}\) is equipped with the inner product \(\langle \cdot, \cdot \rangle\) and the norm \(\| \cdot \|\), respectively,
\[
\|(u,v)\| = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega, \quad \|u\| = |u|_{H^1} = \|(u,v)\|^{1/2}.
\]

The weak formulation of (1.2) reads: To find \(u \in \mathcal{V}\) such that
\[
a_\epsilon(u,v) = F(v), \ \forall v \in \mathcal{V},
\]
(3.4)
where
\[
a_\epsilon(u,v) = \epsilon\|(u,v)\| - (u_y, v) + (p(u), v), \ F(v) = (f,v).
\]
(3.5)

We now define the finite elements space \(V_h \subset \mathcal{V}\) in which we seek an approximate solution:
\[
V_h = \left\{ \sum_{i=0}^{M} c_i \phi_i(x)\psi_0(y) + \sum_{i=0}^{M} \sum_{j=1}^{N-1} c_{ij} \phi_i(x)\psi_j(y) \mid c_{ij} \in \mathbb{R}, \ c_{0j} = c_{Mj} \right\},
\]
(3.6)
where \(\psi_0\) is the BLE as in (3.2) and \(\phi_i, \psi_j\) are the bilinear elements (\(Q_1\)-elements); notice that the number of its basis elements is \(MN\). Then the finite element scheme is: To find \(u_h \in V_h\) such that
\[
a_\epsilon(u_h, v_h) = F(v_h), \ \forall v_h \in V_h.
\]
(3.7)

Thanks to the assumption (2.1), we can verify the existence and uniqueness of the solutions of Eq. (3.4), (3.7) (see e.g. [T97] or see similarly Lemma 5.1 below where we use the monotone operators of Browder-Minty-Lions-Stampacchia type).

We will show that the BLE \(\psi_0^*\) essentially absorbs the \(H^2\)-singularity of \(u^\epsilon\) (see (3.10) below). We then derive the interpolation inequalities as in Lemma 3.1 below under the assumption of the smallness of \(\epsilon\) (see (3.11) below) using classical interpolation results (see [JT05b], [S73]).

From Theorem 2.1 with \(n = 1\), since the \(u^j\) are smooth and independent of \(\epsilon\), we find that
\[
\|u^\epsilon - \theta^0 - \epsilon \theta^1\|_{H^2} \leq \kappa.
\]
Using the explicit expressions for \(\theta^0, \theta^1\) as in (2.10), (2.16), we write
\[
-\theta^0 - \epsilon \theta^1 = g(x) \exp(-y/\epsilon) - \epsilon \eta(x,y) + e.s.t.,
\]
(3.9a)
where \( g(x) = u^0(x,0) - \epsilon C_1(x), \eta(x,y) = \theta_1^y = -\int_0^\infty e^{-\epsilon t} \int_0^1 e^{r_0(r_0 + \theta_t - r_0^0)} ds dt \) (see (2.16)). We then easily verify that \( g, \eta \) are smooth and \( L_1 \)-periodic in \( x \) and estimated as: \( |g(x)|_{H^2(0,L_1)} \leq \kappa \), and for \( l, m \geq 0, 0 < \epsilon < 1, \)

\[
(3.12b) \quad \left| \frac{\partial^{m+l} y}{\partial x^m \partial y^l} \right| \leq \kappa \epsilon^{-l} \exp\left(-\frac{\epsilon^y}{\epsilon}\right) \left| \frac{\partial^{m+l} y}{\partial x^m \partial y^l} \right|_{L^2(\Omega)} \leq \kappa \epsilon^{-l+\frac{m}{2}}.
\]

Hence, setting \( \eta^*(x,y) = \epsilon (\eta(x,y) - \eta(x,0)(1-y) - \eta(x,1)y) \in \mathcal{V} \), we infer from (3.8) using (3.1) that:

\[
(3.10) \quad \|u^\epsilon - g(x)\psi^*_0(y) - \eta^*(x,y)\|_{H^2} \leq \kappa.
\]

We then observe that there exist \( c_j \in \mathbb{R} \) such that \((1-y/h_2)\chi_{[0,h_2]}(y) + \sum_{j=1}^{N-1} c_j \psi_j(y) = 1 - y \). Assuming that \( \epsilon \) is sufficiently small:

\[
(3.11) \quad -\epsilon \ln \epsilon \leq \frac{2}{3} h_2, \quad \epsilon \leq \kappa h_2,
\]

and setting \( \Psi_0(y) = \psi_0(y) + \sum_{j=1}^{N-1} c_j \psi_j(y) \), it is then not hard to verify that for \( m = 0, 1, \)

\[
(3.12a) \quad |\Psi_0|_{H^m(0,L_2)} \leq \kappa \epsilon^{-\frac{m}{2}},
\]

\[
(3.12b) \quad |\psi^*_0 - \Psi_0|_{H^m(0,L_2)} \leq \kappa h_2^{2-m},
\]

\[
(3.12c) \quad |\eta^* + \epsilon \eta(x,0)|_{H^m(0,L_2)} \leq \kappa \epsilon^\frac{2}{2-m};
\]

see [J06a], [JT05c]. We thus deduce the following interpolation inequalities:

**Lemma 3.1.** Assume that (3.11) holds. Then there exists an interpolant \( \tilde{u}_h \in V_h \) such that

\[
(3.13a) \quad \|u^\epsilon - \tilde{u}_h\|_{L^2(\Omega)} \leq \kappa (h^2 + \epsilon^\frac{2}{2}),
\]

\[
(3.13b) \quad \|u^\epsilon - \tilde{u}_h\|_{H^1(\Omega)} \leq \kappa (h + h_1^2 \epsilon^{\frac{1}{2}} + \epsilon^\frac{1}{2}).
\]

**Proof.** By the classical interpolation theory, see e.g. [Cia78], [S73], setting \( \tilde{u}_* = u^\epsilon - g(x)\psi^*_0(y) - \eta^*(x,y) \in \mathcal{V} \) and using (3.10), we find that there exist \( \Pi \tilde{u}_*, \Pi g \) and \( \Pi \eta \) such that for \( m = 0, 1, \)

\[
(3.14) \quad I_1(m) := \|u^\epsilon - \Pi \tilde{u}_*\|_{H^m(\Omega)} \leq \kappa h_2^{2-m}\|\tilde{u}_*\|_{H^2(\Omega)} \leq \kappa h_2^{2-m},
\]
Lemma 2.1 and the Sobolev imbedding inequality, for

we then need to majorize the norm

\( \| \psi \|_{H^m(\Omega)} \)

from (3.12) we thus deduce that

\[
I_2(m) := \| g(x) \psi'_0(y) - \Pi_x g(x) \Psi_0(y) \|_{H^m(\Omega)}
\leq \kappa \| g(x) \psi'_0(y) - \Psi_0(y) \|_{H^m(\Omega)} + \kappa \| g(x) - \Pi_x g(x) \Psi_0(y) \|_{H^m(\Omega)}
\leq \kappa \left\{ \begin{array}{ll}
|g|_{L^2(\Omega)} \| \psi'_0 - \Psi_0 \|_{L^2(\Omega)} + & |g - \Pi_x g|_{L^2(\Omega)} \| \Psi_0 \|_{L^2(\Omega)} \\
|g|_{L^2(\Omega)} \| \psi'_0 - \Psi_0 \|_{H^1(\Omega)} + & |g|_{H^1(\Omega)} \| \psi'_0 - \Psi_0 \|_{L^2(\Omega)} \\
+ |g - \Pi_x g|_{H^1(\Omega)} \| \Psi_0 \|_{H^1(\Omega)} + & |g - \Pi_x g|_{H^1(\Omega)} \| \Psi_0 \|_{L^2(\Omega)}
\end{array} \right.
\]

\( \leq \kappa \left\{ \begin{array}{ll}
\frac{h^2}{h + h^2 \epsilon^{-\frac{1}{2}}} & \text{for } m = 0, \\
\epsilon^2 + ch^2 & \text{for } m = 1.
\end{array} \right. \)

Hence setting \( \tilde{u}_h = \Pi \psi(x) + \Pi_x g(x) \Psi_0(y) - \epsilon \Pi_x \eta(x, 0) \Psi_0(y) \), which belongs to the finite elements space \( V_h \), we deduce that

\[
\| u - \tilde{u}_h \|_{H^m(\Omega)} \leq I_1(m) + I_2(m) + I_3(m);
\]

this proves the lemma.

□

Using the interpolation inequalities as in Lemma 3.1, we are now able to deduce the following approximation errors.

**Lemma 3.2.** Assume that (3.11) holds and let \( u = u^* \) be the solution of (3.4) and \( u_h \) the solution of (3.7). Then there exists a positive constant \( \kappa \) independent of \( \epsilon \) such that

\[
\| u - u_h \|_{H^1(\Omega)} \leq \kappa (h^2 \epsilon^{-1} + \epsilon^\frac{1}{2}).
\]

**Proof.** Subtracting (3.7) from (3.4), we find that

\[
\epsilon((u - u_h, v_h)) - ((u - u_h)_y, v_h) + (p(u) - p(u_h), v_h) = 0, \quad \forall v_h \in V_h.
\]

Hence we can write: for an interpolant \( \tilde{u}_h \) as in Lemma 3.1 and for all \( v_h \in V_h \),

\[
\epsilon((u - \tilde{u}_h, v_h)) - ((u - \tilde{u}_h)_y, v_h) + (p(u) - p(\tilde{u}_h), v_h) = \epsilon((u - \tilde{u}_h, v_h)) - ((u - \tilde{u}_h)_y, v_h) + (p(u) - p(\tilde{u}_h), v_h).
\]

Setting \( v_h = u - \tilde{u}_h \in V_h \), thanks to (2.1), we find that

\[
\epsilon |u - \tilde{u}_h|_{H^1} \leq \epsilon |u - \tilde{u}_h|_{H^1} |u_h - \tilde{u}_h|_{H^1} \]

\[
+ |u - \tilde{u}_h|_{L^2} |u_h - \tilde{u}_h|_{H^1} + |p(u) - p(\tilde{u}_h)|_{L^2} |u_h - \tilde{u}_h|_{L^2}.
\]

We then need to majorize the norm \( |p(u) - p(\tilde{u}_h)|_{L^2} \). We firstly notice that from Lemma 2.1 and the Sobolev imbedding inequality, for \( n = 1 \),

\[
|u|_{L^\infty} \leq |u|_{L^\infty} + |\theta|_{L^\infty} + |w|_{L^\infty} \leq |u|_{L^\infty} + |\theta|_{L^\infty} + \kappa |w|_{H^1} \leq \kappa.
\]
Recalling that \( \tilde{u}_h = \Pi \tilde{u} + \Pi_x g(x) \Psi_0(y) - \epsilon \Pi_x \eta(x,0) \Psi_0(y) \), since \(|\tilde{u}'|_{H^2(\Omega)} \leq \kappa, |g(x)|_{H^2(0,1)} \leq \kappa, |\eta(x,0)|_{H^2(0,1)} \leq \kappa\), we easily find that \(|\tilde{u}_h|_{L^\infty} \leq \kappa\). By the mean value theorem, we thus have
\[
|p(u) - p(\tilde{u}_h)| \leq |p'((1-\rho)u + \rho \tilde{u}_h)||u - \tilde{u}_h| \leq \kappa |u - \tilde{u}_h|,
\]
where \( \rho = \rho(x) \) is between 0 and 1. Hence \(|p(u) - p(\tilde{u}_h)|_{L^2} \leq \kappa |u - \tilde{u}_h|_{L^2} \). By the Cauchy-Schwarz and the Poincaré inequalities applied to (3.21), we thus find
\[
\epsilon |u_h - \tilde{u}_h|_{H^1}^2 \leq \kappa \epsilon |u - \tilde{u}_h|_{H^1}^2 + \kappa \epsilon^{-1} |u - \tilde{u}_h|_{L^2}^2.
\]
Since \(|u - u_h|_{H^1} \leq |u - \tilde{u}_h|_{H^1} + |u_h - \tilde{u}_h|_{H^1} \), from the interpolation inequalities as in Lemma 3.1 the lemma follows.

**Remark 3.1.** By Lemma 3.2, to have an effective finite element scheme (3.7) we require the mesh size \( h \) to be \( o(\epsilon^{1/2}) \) in the \( H^1 \)-space. Furthermore, if \( p(\xi) = q(\xi) + \lambda \xi, \lambda > 0 \), and \( q(\xi_1) - q(\xi_2)\) \((\xi_1 - \xi_2) \geq 0 \), we can deduce the error in the weighted energy norm, \( \| \cdot \| = \epsilon^{1/2} \cdot | \cdot |_{H^1} + | \cdot |_{L^2} \). We notice that the left-hand side of (3.21) becomes \( \epsilon |u_h - \tilde{u}_h|_{H^1}^2 + \lambda |u_h - \tilde{u}_h|_{L^2}^2 \) and then similarly as in the proof of Lemma 3.2, we find that
\[
\|u - u_h\|_\epsilon \leq \kappa (h^2 \epsilon^{-1/2} + \epsilon).
\]
In this case, we require the effective mesh size \( h \) to be \( o(\epsilon^{1/2}) \) in the weighted energy norm.

4. **Numerical Simulations**

4.1. **Pseudo-arclength continuation methods.** Since using the BLE we absorbed the singularities due to the boundary layers of the exact solutions \( u^* \) (resp. \( u_h \)) in (3.4) (resp. (3.7))\(^1\), we expect that our discretized systems (3.7) can be solved via a classical iterative method. Here we employ the technique of pseudo-arclength continuation (see e.g. [D00], [K87], [AG90]). We write (3.7) in the matrix form but we multiply \( F(u_h) \) by \( \lambda \) for \( 0 \leq \lambda \leq 1 \),

\[
F(x, \lambda) = G(x) - \lambda f = 0,
\]
where \( x = [\cdots, x_{ij}, \cdots]^T \), \( x_{0j} = x_{Mj} \), \( f = [\cdots, (f, \phi_k \psi_l), \cdots]^T \) and \( G(x) = [\cdots, G(x)_{kl}, \cdots]^T \) with
\[
G(x)_{kl} = \sum_{i,j} x_{ij} \{ \epsilon (\partial_i \psi_j, \partial_i \psi_l) - \epsilon (\phi_{ij}, \phi_{ij}) \} + \left( \int_{\xi} \sum_{i,j} x_{ij} \phi_{ij} \psi_l \right),
\]
for \( 0 \leq i, k \leq M, 0 \leq j, l \leq N - 1 \); the function \( \psi_0 \) is the BLE, \( \phi_i, \psi_j \) are \( Q_1 \)-elements (piecewise linear) as in the finite element space \( V_h \), (3.6), and the \( \lambda \) represent the intensity of a forcing data \( f \). Note that \( F \) is a smooth mapping from \( \mathbb{R}^{MN+1} \) into \( \mathbb{R}^M \) and \( M \) (resp. \( N \)) is the number of finite elements in the \( x \)-(resp. \( y \)-)direction.

To solve the nonlinear algebraic system (4.1), we may consider the Newton’s iterative method for \( \lambda = 1 \), see e.g. [P03], but in practice we have to choose the

\(^1\)Recall that if we do not appropriately handle the singularities, the solutions of the discretized systems display wild oscillations or a high instability, see [JT05]-[JT05c].
Our goal is thus to find the point \( \mathbf{x}(s) \) when we reach \( \lambda(s) = 1 \) along the curve \( \gamma \). We firstly choose \((\mathbf{x}^1, \lambda^1)\) which is close to the point \((\mathbf{x}_0, \lambda_0) \in \gamma(s)\); the point \((\mathbf{x}^1, \lambda^1)\) will play a role of a good initial guess: for a step \( \Delta s > 0 \),

\[
\begin{align*}
\mathbf{x}^1 &= \mathbf{x}_0 + \Delta s \dot{\mathbf{x}}_0 \\
\lambda^1 &= \lambda_0 + \Delta s \dot{\lambda}_0,
\end{align*}
\]

where the tangent vector \((\dot{\mathbf{x}}_0, \dot{\lambda}_0) := \dot{\gamma}(s_0) = (\dot{\mathbf{x}}(s_0), \dot{\lambda}(s_0)) \) (\( \cdot \) stands for \( d/ds \)) and it is normalized, i.e.,

\[
|\dot{\gamma}(s_0)|^2 = \dot{\mathbf{x}}_0^T \dot{\mathbf{x}}_0 + \dot{\lambda}_0^2 = 1.
\]

To find such tangent vector \( \dot{\gamma}(s_0) \), we just differentiate Eq. (4.2a) with respect to \( s \) evaluated at \( s = s_0 \):

\[
\begin{bmatrix}
D \Phi & \Phi \lambda \\
0 & 1
\end{bmatrix} \dot{\gamma}(s_0) = \begin{bmatrix}
DG & -\mathbf{f} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\mathbf{x}_0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix};
\]

to close the system, here we fixed \( \dot{\lambda}_0 = 1 \). Hence, we only need to solve \( DG \dot{x}_0 = \mathbf{f} \) and then we normalize \( \dot{\gamma}(s_0) \), i.e. \( \dot{\gamma}(s_0)/|\dot{\gamma}(s_0)| = (\dot{x}_0, 1)(\dot{x}_0^T \dot{x}_0 + 1)^{-1/2} \) which satisfy (4.3c). Note that since \( \dot{\lambda}_0 > 0 \), we can increase \( \lambda \) by \( \Delta s \dot{\lambda}_0 \) with an appropriate small \( \Delta s > 0 \) and choose the initial guess \((\mathbf{x}^1, \lambda^1)\) as in (4.3) (see Figure 2).
We now introduce a plane \( \mathcal{N} \), passing through \((x^1, \lambda^1)\), perpendicular to the tangent vector \( \dot{\gamma}(s_0) \):

\[
\mathcal{N} = \{ \varphi(x, \lambda) = 0 \},
\]

where

\[
\varphi(x, \lambda) = x_0^T(x - x^1) + \dot{\lambda}_0(\lambda - \lambda^1)
\]

\[
= (\text{thanks to } (4.3))
= \dot{x}_0^T(x - x_0) + \dot{\lambda}_0(\lambda - \lambda_0) - \Delta s.
\]

Since we absorbed the boundary layer singularities as we have seen in the previous sections, \( \Phi(x, \lambda) \) is non-stiff smooth and thus, with \( \Delta s \) small, we may regard that the point \((x^1, \lambda^1) \in \mathcal{N}\) is a good initial guess (very close) to the solution \((\bar{x}, \bar{\lambda})\) of Eq. (4.1) which we now seek in the plane \( \mathcal{N} \) (see Figure 2). To find \((\bar{x}, \bar{\lambda})\), we just apply the Newton’s iterative method (see e.g. [P03]) to the augmented system

\[
(4.6) \quad \bar{\Phi}(x, \lambda) = \begin{bmatrix} \Phi(x, \lambda) \\ \varphi(x, \lambda) \end{bmatrix} = 0.
\]

More precisely, setting \((x^1, \lambda^1)\) as in (4.3), we iteratively solve for \(x^{k+1}, \lambda^{k+1}\) in the following linear system:

\[
(4.7a) \quad D\bar{\Phi}(x^k, \lambda^k)\Delta x^k = -\bar{\Phi}(x^k, \lambda^k),
\]

where

\[
(4.7b) \quad D\bar{\Phi}(x^k, \lambda^k) = \begin{bmatrix} D\Phi(x^k, \lambda^k) & \Phi(x^k, \lambda^k) \\ \dot{x}_0^T & \dot{\lambda}_0 \end{bmatrix},
\]

\[
(4.7c) \quad \Delta x^k = \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix} = \begin{bmatrix} x^{k+1} - x^k \\ \lambda^{k+1} - \lambda^k \end{bmatrix},
\]

\[
(4.7d) \quad -\bar{\Phi}(x^k, \lambda^k) = \begin{bmatrix} -\Phi(x^k, \lambda^k) \\ \Delta s - \dot{x}_0^T(x^k - x_0) - \dot{\lambda}_0(\lambda^k - \lambda_0) \end{bmatrix}.
\]

In our problem (4.1), we can obtain the Jacobian matrix \(D\bar{\Phi}(x^k, \lambda^k)\) as follows: for \(x^k = [\cdots, x_{ij}, \cdots]^T\),

\[
(4.8) \quad \begin{bmatrix} \epsilon(\phi_i \psi_j, \phi_k \psi_l) & (\phi_i \psi_{jy}, \phi_k \psi_l) & (p' \psi_{mn} \phi_{mn} \psi_n) \phi_i \psi_j, \phi_k \psi_l) & -f \\ \dot{x}_0^T \end{bmatrix}.
\]

Once we find the solution \((\bar{x}, \bar{\lambda})\) on the \( \gamma \) (see Figure 2), setting \((x_0, \lambda_0) = (\bar{x}, \bar{\lambda})\) we continue the same procedures until we reach \( \lambda = 1 \); indeed, since \( \lambda_0 > 0 \), we can increase \( \lambda \) at each continuation and eventually reach \( \lambda = 1 \).

4.2. Examples. In this section we present some numerical simulations utilizing the boundary layer element \( \psi_0(y) \) (see Figure 1) and the classical elements, namely hat functions \((Q_1\text{-elements})\). For the numerical testing, in Tables 1 - 2 on a domain \( \Omega = (0, 2) \times (0, 1) \) we tried a given solution \( u \) as below:

\[
u = (1 - e^{-y/\epsilon})(1 - y) \sin(\pi x) \text{ with } p(u) = u^3;
\]

the function \( f \) is obtained from this \( u \). Using MAPLE we have derived the exact formula for the integrations for the nonlinear terms (related to the polynomial \( p(u) \) as in (4.1b), (4.8)). However since the calculations are involved, we also tried the
numerical integrations for them, which are more practical and useful but we have to allow some numerical errors. We employed the composite Simpson’s rule for the double integrations (see e.g. [P03]). For the system solver of (4.1), as we just described, we use the pseudo-arclength continuation method. The integrations related to $p$ using both the composite Simpson’s rule and the exact formula are shown in Tables 1 - 2; the $L^2$- and $L^\infty$- errors, i.e. $|u - u_h|_{L^2(\Omega)}$, $|u - u_h|_{L^\infty(\Omega)}$, are provided.

In Figure 3, we tested a Dirichlet boundary value problem:

$$
\begin{cases}
-\epsilon \Delta u^\epsilon + p(u^\epsilon) = f & \text{in } \Omega = (0, 1) \times (0, 1), \\
u^\epsilon = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(4.10)

where $f = e^y(1 - e^{-x})(1 - e^{-(1-x)})$ and $p(u) = u, u^3, u^5$. In this case, unlike the problem (1.2) which is periodic in $x$, we expect the discrepancies at $x = 0, 1$ between the boundary conditions $u^\epsilon = 0$ and the limit solution $u^0$ to account for some (nonlinear) boundary layers at $x = 0, 1$. But in Figure 3 we do not observe any boundary layers at $x = 0, 1$ which affect our numerical simulations; since $f(0, y) = f(1, y) = 0$, we expect that they are weakened. Indeed, for simplicity, assuming that $u^0 \in C^1(\bar{\Omega})$, from (2.5a) we infer that for $x = 0$ (it is similar for $x = 1$), $-u^0_y(0, y) + p(u^0)(0, y) = 0$. Hence from (2.1) we find that

$$
\frac{d(u^0)^2}{dy} = 2u^0 p(u^0) \geq 0,
$$

(4.11)

and thus by the boundary condition (2.5b), we obtain $u^0(0, y) = 0$. This implies that there are no discrepancies at $x = 0, 1$ of $u^0$ with the boundary condition of $u^\epsilon$ but the outer solutions $u^j, j \geq 1$, may have discrepancies at those boundaries. Hence the boundary layers at $x = 0, 1$ may appear at the order of $O(\epsilon^j)$, $j \geq 1$, in the asymptotic analysis and these boundary layers are mild. If $f(0, y) \neq 0$ or $f(1, y) \neq 0$, as in the linear case of [JT05a], the boundary layers are outstanding at $x = 0, 1$. Here, in Figure 3, we used the BLE $\psi_0$ in the finite element space as before. It is noteworthy that the solutions are more weighted to the side $y = 0$ as the order in $p(u)$ increases.

5. Existence and uniqueness of $C^\infty$ solutions $u^j$ for the equations (2.5)

We now study the existence, uniqueness and regularity of periodic solutions $u^j$ for the equations (2.5). In this section, by regularity of $u^j$, we mean the regularity of its periodic extension, which implies (means), smooth matchings at 0 and $L_1$.

We first notice that the equation for $u^0$ is a nonlinear ODE while the equations for the $u^j, j \geq 1$ are linear ODEs, since the $r^j_u, j \geq 1$ are polynomials of order 1 in $u^j$ with coefficients which are expressions of $u^0, \ldots, u^{j-1}$; see (2.2), (2.3). Hence it is easily seen that if the periodic extension of $u^0$ is smooth then so are the periodic extensions of the $u^j$.

Thus we first study the existence, uniqueness and regularity of periodic solutions for the following nonlinear ODE problem corresponding to (2.5) with $v = u^0$:

$$
\begin{cases}
-v_y + p(v) = f & \text{in } \Omega, \\
v(x, L_2) = 0, & x \in (0, L_1).
\end{cases}
$$

(5.1)
Here \( f : \Omega \rightarrow \mathbb{R} \) is as smooth as needed as well as its periodic extension \( \hat{f} \) to \( \bar{\Omega} \) and \( p : \mathbb{R} \rightarrow \mathbb{R} \) is smooth and monotone increasing as in (2.1).

Let \( A : D(A) \rightarrow L^2(\Omega) \) be defined by \( A(v) = -v_y + p(v) \), where \( D(A) := \{ v \in L^{2s}(\Omega) \text{ such that } v_y \in L^2(\Omega) \text{ and } v(\cdot, L_2) = 0 \} \subset L^2(\Omega) \). The equation (5.1) then reads:

To find \( v \in D(A) \) such that

\[
A(v) = f \quad \text{in } \Omega,
\]

In order to prove the existence of (5.1), we will consider the perturbed problems (5.3) below on \( \Omega \) - which are different than (1.2) - and then prove a convergence result for these perturbed solutions to a solution of (5.1), which shows the existence of a solution to (5.1) on \( \Omega \). The uniqueness of solution of (5.1) comes from the property (2.1).

Next, we extend the solution to the whole channel \( \bar{\Omega} \) and prove that it satisfies the equation (5.1) extended to \( \bar{\Omega} \) and that this periodic extension has \( C^\infty \) regularity. This also shows the periodicity of the solution of the limit problem since \( \bar{\Omega} \) is periodic.

To begin with the proof of the existence of solution for (5.1), we consider the following perturbed problems in the domain \( \Omega \) for each \( \epsilon > 0 \):

\[
\begin{cases}
-\epsilon \Delta v^\epsilon - v_y^\epsilon + p(v^\epsilon) = f & \text{in } \Omega, \\
\partial v^\epsilon(y, 0) = 0, \quad \partial v^\epsilon(x, L_2) + v^\epsilon(x, L_2) = 0, \quad \forall x \in (0, L_1), \\
v^\epsilon(0, y) = v^\epsilon(L_1, y), \quad \partial x v^\epsilon(0, y) = \partial x v^\epsilon(L_1, y), \quad \forall y \in (0, L_2).
\end{cases}
\]  

(5.3)

The variational setting for these perturbed problems is as follows: \( V = \{ v \in H^1(\Omega) \cap L^{2s}(\Omega) \mid v(0, \cdot) = v(L_1, \cdot) \} \)\(^2\) endowed with the norm \( \| v \|_V = \| \nabla v \|_{L^2(\Omega)} + \| v \|_{L^{2s}(\Omega)} \). The existence and uniqueness result for the solutions of (5.3) is given in the following lemma:

**Lemma 5.1.** For each \( \epsilon > 0 \), there exists a unique weak solution \( v^\epsilon \) in \( V \) for the perturbed problem (5.3).

**Proof.** We first observe that the weak formulation of (5.3) is of the form:

To find \( v \in V \) such that

\[
\begin{align}
\mathbf{a}_\epsilon(v, \varphi) &= l(\varphi), \quad \forall \varphi \in V, \\
\mathbf{a}_\epsilon(v, \varphi) &= \epsilon \int_{\Omega} \nabla v \cdot \nabla \varphi + \int_{\Omega} v \varphi_y + \int_{y=0} v \varphi d\Gamma + \int_{\Omega} p(v) \varphi, \\
l(\varphi) &= \int_{\Omega} f \varphi.
\end{align}
\]  

(5.4a)\(^3\) (5.4b)\(^4\) (5.4c)

To see this, we multiply (5.3) by \( \varphi \in V \) and integrate over \( \Omega \); using Stokes formula and the boundary conditions for \( v^\epsilon \) and \( \varphi \), we find (5.4a). Classically, one can show that, conversely, a regular solution of (5.4a) is a solution of (5.3).

Since \( \mathbf{a}_\epsilon(\cdot, \cdot) \) is linear in the second variable but not linear in the first variable, the Lax–Milgram Theorem can not be applied to prove the existence of solution to

\(^2\)In our case, space dimension \( n = 2 \), \( V = \{ v \in H^1(\Omega) \mid v(0, \cdot) = v(L_1, \cdot) \} \) by the Sobolev embedding theorem. The space \( L^{2s}(\Omega) \) is mentioned for future extensions to higher dimensions.
the weak formulation (5.4a). Instead, we will utilize the following result from the theory of monotone operators of Browder-Minty-Lions-Stampacchia type, see e.g. [L69], which can be stated as follows: \( A_c : V \to V' \) is a surjective map as long as the following conditions hold

\[(5.5) \quad A_c \text{ is bounded and hemicontinuous on } V,\]
\[(5.6) \quad A_c \text{ is monotone, i.e. } \langle A_c u - A_c v, u - v \rangle \geq 0, \forall u, v \in V,\]
\[(5.7) \quad A_c \text{ is coercive in the sense that } \lim_{\|v\|_V \to \infty} \frac{\langle A_c v, v \rangle}{\|v\|_V} = \infty.\]

In order to apply this result, we define \( A_c : V \to V' \) as follows: for each \( v \in V \) we define \( A_c(v) \in V' \) by setting

\[(5.8) \quad \langle A_c(v), \varphi \rangle = a_c(v, \varphi), \forall \varphi \in V.\]

Our goal now is to verify that the definition (5.8) makes sense, i.e. \( A_c(v) \in V', \forall v \in V \), and the validity of the conditions (5.5) - (5.7). We will need the following fact that can be deduced from the hypothesis (1.1):

\[(5.9) \quad |p(\xi)| \leq c_1 |\xi|^{2s-1} + c_2, \forall \xi \in \mathbb{R},\]

where \( c_1, c_2 > 0 \) are constants which may depend on \( p \) but not on \( \xi \). Indeed, from (1.1) there exists \( c_0, c_1 > 0 \) such that \( |p(\xi)| \leq c_1 |\xi|^{2s-1} \) for \( |\xi| \geq c_0 \). For \( |\xi| \leq c_0 \), we can easily find \( c_2 > 0 \) such that \( |p(\xi)| \leq c_2 \).

To see that the definition of \( A_c(v) \) makes sense i.e. \( A_c(v) \in V' \) for fixed \( v \in V \), we observe that \( \langle A_c(v), \cdot \rangle \) is linear since \( a_c(v, \cdot) \) is linear, and we prove that \( A_c(v) \) is continuous (bounded) in \( \varphi \) as follows:

\[
|\langle A_c(v), \varphi \rangle| \leq \epsilon |\nabla v|_{L^2(\Omega)} |\nabla \varphi|_{L^2(\Omega)} + |v|_{L^2(\Omega)} |\varphi|_{L^2(\Omega)}
\]
\[
+ |v(\cdot, 0)|_{L^2_0(0, L_1)} |\varphi(\cdot, 0)|_{L^2_0(0, L_1)} + \int_{\Omega} p(v) \varphi
\]

for all \( \varphi \in V \). We then bound the terms on the right-hand side of the above inequality as follows:

\[
\begin{cases}
\epsilon |\nabla v|_{L^2(\Omega)} |\nabla \varphi|_{L^2(\Omega)} \leq \epsilon \|v\|_V \|\varphi\|_V, \\
|v|_{L^2(\Omega)} |\varphi|_{L^2(\Omega)} \leq |v|_{L^2(\Omega)} |\nabla \varphi|_{L^2(\Omega)} \leq \|v\|_V \|\varphi\|_V, \\
|v(\cdot, 0)|_{L^2_0(0, L_1)} |\varphi(\cdot, 0)|_{L^2_0(0, L_1)} \leq c_1 \|v\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)} \leq c \|v\|_V \|\varphi\|_V, \\
\int_{\Omega} p(v) \varphi \leq c_1 |v|_{L^{2s-1}(\Omega)}^{2s-1} |\varphi|_{L^2(\Omega)} + c_2 \sqrt{\Omega} |\varphi|_{L^2(\Omega)}.
\end{cases}
\]

Here we have used the estimate (5.9) for the last inequality. Thus we have

\[(5.10) \quad |\langle A_c(v), \varphi \rangle| \leq \epsilon (\epsilon, p)(\|v\|^{2s-1}_V + 1) \|\varphi\|_V, \forall \varphi \in V.\]

Therefore \( A_c(v) \) belongs to \( V' \) and its norm in \( V' \) satisfies

\[(5.11) \quad \|A_c(v)\|_{V'} = \sup_{\|\varphi\|_V \leq 1} |\langle A_c(v), \varphi \rangle| \leq \epsilon (\epsilon, p)(\|v\|^{2s-1}_V + 1).\]

To verify hemicontinuity (5.5): The boundedness of \( A_c \) follows from (5.11). We now check the hemicontinuity, i.e. the function \( \lambda \to \langle A_c(u + \lambda v), \varphi \rangle \) is continuous from \( \mathbb{R} \) to \( \mathbb{R} \) for fixed \( v, \varphi \in V \). We have

\[
\langle A_c(u + \lambda v), \varphi \rangle = \epsilon (\nabla (u + \lambda v), \nabla \varphi)_{L^2(\Omega)} + (u(\cdot, 0) + \lambda \varphi(\cdot, 0), \varphi(\cdot, 0))_{L^2_0(0, L_1)}
\]
\[
+ (u + \lambda v, \varphi)_{L^2(\Omega)} + \langle p(u + \lambda v), \varphi \rangle_{L^{2s-1}(\Omega), L^2(\Omega)}.
\]
Lemma 5.2. The solutions $v^\varepsilon$ of (5.4a) satisfy
\begin{equation}
\begin{cases}
\varepsilon^2 \|\nabla v^\varepsilon\|_{L^2(\Omega)} \leq C, \\
|v^\varepsilon(\cdot, 0)|_{L^2(0, L_2)} \leq C, |v^\varepsilon(\cdot, L_2)|_{L^2(0, L_1)} \leq C, \\
\|v^\varepsilon\|_{L^\infty(\Omega)} \leq C,
\end{cases}
\end{equation}
for a positive constant $C$ independent of $\varepsilon$. 

where $\langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle_{B,B'}$ denote respectively the inner product of the Hilbert space $H$ and the dual pair between the Banach space $B$ and its dual $B'$. Hence it is easy to see that the first three terms are continuous in $\lambda$. For the last term, due to (5.9) the Lebesgue Dominated Convergence Theorem applies and the hemi-continuity for $A_\varepsilon$ then follows.

To verify (5.6): Let $u, v \in V$, we have by straightforward calculations and (2.1) that
\begin{equation*}
\langle A_\varepsilon(u) - A_\varepsilon(v), u - v \rangle = \langle A_\varepsilon(u), u - v \rangle - \langle A_\varepsilon(v), u - v \rangle
= \varepsilon \int_\Omega |\nabla (u - v)|^2 + \frac{1}{2} \int_{y=0} (u - v)^2 + \frac{1}{2} \int_{y=L_2} (u - v)^2 + \int_\Omega (p(u) - p(v))(u - v)
\geq 0.
\end{equation*}

To verify (5.7): Let $v \in V$; again by straightforward calculations, we find
\begin{equation*}
\langle A_\varepsilon(v), v \rangle = \varepsilon \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_{y=L_2} v^2 d\Gamma + \frac{1}{2} \int_{y=0} v^2 d\Gamma + \int_\Omega p(v)v
\geq \varepsilon \int_\Omega |\nabla v|^2 + \int_\Omega p(v)v.
\end{equation*}
Using (1.1) and Young’s inequality ($a^{2s} \geq a^2 - (s - 1)s^{-s/(s-1)}$), we see that
\begin{equation*}
\langle A_\varepsilon(v), v \rangle \geq \varepsilon \int_\Omega |\nabla v|^2 + \gamma \int_\Omega u^{2s} - c_I \geq \frac{\min(\varepsilon, \gamma)}{2} \|v\|_{V'}^2 - c_I - \frac{s - 1}{s^{s/(s-1)}}.
\end{equation*}
Thus (5.7) holds and the existence of weak solution to (5.3) then also follows. We now prove the uniqueness of the weak solution. Suppose that $v_1^\varepsilon, v_2^\varepsilon$ are two solutions to (5.4a); then
\begin{equation*}
\varepsilon \int_\Omega |\nabla (v_2^\varepsilon - v_1^\varepsilon)|^2 + \frac{1}{2} \int_{y=0} (v_2^\varepsilon - v_1^\varepsilon)^2 d\Gamma + \frac{1}{2} \int_{y=L_2} (v_2^\varepsilon - v_1^\varepsilon)^2 d\Gamma
\geq \int_\Omega (p(v_2^\varepsilon) - p(v_1^\varepsilon)) (v_2^\varepsilon - v_1^\varepsilon) = 0.
\end{equation*}
Hence with (2.1):
\begin{equation*}
\varepsilon \int_\Omega |\nabla (v_2^\varepsilon - v_1^\varepsilon)|^2 \leq 0,
\end{equation*}
i.e. $v_2^\varepsilon - v_1^\varepsilon$ is a constant. Due to the boundary condition $\varepsilon v_y(x, L_2) + \varepsilon^x(x, L_2) = 0$ we conclude that $v_1^\varepsilon = v_2^\varepsilon$ a.e. This finishes the proof of the lemma. \hfill \square

We need the following a priori estimates on the perturbed solutions $v^\varepsilon$ before passing to the limit $\varepsilon \to 0$ for the existence of the limit problem:

Lemma 5.2. The solutions $v^\varepsilon$ of (5.4a) satisfy
\begin{equation}
\begin{cases}
\varepsilon^2 \|\nabla v^\varepsilon\|_{L^2(\Omega)} \leq C, \\
|v^\varepsilon(\cdot, 0)|_{L^2(0, L_2)} \leq C, |v^\varepsilon(\cdot, L_2)|_{L^2(0, L_1)} \leq C, \\
\|v^\varepsilon\|_{L^\infty(\Omega)} \leq C,
\end{cases}
\end{equation}
for a positive constant $C$ independent of $\varepsilon$. 

Proof. We first prove (5.12)1–2. Taking \( \varphi = v^\varepsilon \) in (5.4a), we find
\[
\epsilon \int_\Omega |\nabla v^\varepsilon|^2 + \frac{1}{2} \int_{y=0} |v^\varepsilon|^2 d\Gamma + \frac{1}{2} \int_{y=L_2} |v^\varepsilon|^2 d\Gamma + \int_\Omega p(v^\varepsilon)v^\varepsilon = \int_\Omega f v^\varepsilon.
\]
We bound the nonlinear term on the left hand side from below and the term on the right hand side from above using (1.1) and the Hölder and Young inequalities:
\[
\begin{aligned}
&\int_\Omega p(v^\varepsilon)v^\varepsilon \geq \gamma l \int_\Omega (v^\varepsilon)^2s - c_l, \\
&\int_\Omega |f v^\varepsilon| = \int_\Omega \left| (s\gamma_l)^{\frac{2}{s}} v^\varepsilon (s\gamma_l)^2 \right| f \leq \gamma l \int_\Omega |v^\varepsilon|^{2s} + \frac{2s-1}{2s(s\gamma_l)^{\frac{2s}{s-1}}} \int_\Omega |f|^{\frac{2s}{s-1}}.
\end{aligned}
\]
Hence we obtain
\[
\epsilon \int_\Omega |\nabla v^\varepsilon|^2 + \frac{1}{2} \int_{y=0} |v^\varepsilon|^2 d\Gamma + \frac{1}{2} \int_{y=L_2} |v^\varepsilon|^2 d\Gamma + \frac{\gamma l}{2} \int_\Omega |v^\varepsilon|^{2s} \leq \frac{2s-1}{2s(s\gamma_l)^{\frac{2s}{s-1}}} \int_\Omega |f|^{\frac{2s}{s-1}} + c_l.
\]
Thus the inequalities (5.12)1 and (5.12)2 hold.
To obtain the bound (5.12)3, we apply the maximum principle argument as follows: let \( C > 0 \) be such that
\[
P(\xi) \geq \sup f, \text{ for all } \xi \geq C.
\]
Taking \( \varphi = (v^\varepsilon - C)^+ \) in (5.4a), and noticing that
\[
\begin{aligned}
&\int_{y=0} v^\varepsilon (v^\varepsilon - C)^+ d\Gamma = \int_{y=0} ((v^\varepsilon - C)^+)^2 d\Gamma + C \int_{y=0} (v^\varepsilon - C)^+ d\Gamma, \\
&\int_\Omega v^\varepsilon (v^\varepsilon - C)^+ \bigg|_{y=L_2} = \frac{1}{2} \int_{y=L_2} ((v^\varepsilon - C)^+)^2 d\Gamma - \frac{1}{2} \int_{y=0} ((v^\varepsilon - C)^+)^2 d\Gamma + C \int_{y=L_2} (v^\varepsilon - C)^+ d\Gamma - C \int_{y=0} (v^\varepsilon - C)^+ d\Gamma,
\end{aligned}
\]
we find
\[
\epsilon \int_\Omega |\nabla (v^\varepsilon - C)^+|^2 + \frac{1}{2} \int_{y=0} ((v^\varepsilon - C)^+)^2 d\Gamma + \frac{1}{2} \int_{y=L_2} ((v^\varepsilon - C)^+)^2 d\Gamma + C \int_{y=L_2} (v^\varepsilon - C)^+ d\Gamma + \int_\Omega (p(v^\varepsilon) - f)(v^\varepsilon - C)^+ = 0.
\]
We know by (2.13) that \( p(v^\varepsilon) \geq \sup f \) on the support of \((v^\varepsilon - C)^+\); hence we have
\[
\epsilon \int_\Omega |\nabla (v^\varepsilon - C)^+|^2 + \frac{1}{2} \int_{y=0} ((v^\varepsilon - C)^+)^2 d\Gamma + \frac{1}{2} \int_{y=L_2} ((v^\varepsilon - C)^+)^2 d\Gamma \leq 0.
\]
Thus we find that
\[
|\nabla (v^\varepsilon - C)^+|_{L^2(\Omega)} = \|(v^\varepsilon - C)^+ (\cdot, 0)\|_{L^2(0, L_1)} = \|(v^\varepsilon - C)^+ (\cdot, 1)\|_{L^2(0, L_1)} = 0,
\]
and this implies that
\[
(v^\varepsilon - C)^+ = 0 \text{ a.e. in } \Omega,
\]
that is
\[
(v^\varepsilon - C)^+ \leq C \text{ a.e. in } \Omega.
\]
Similarly, we can bound \( v^\varepsilon \) from below independently of \( \epsilon \). Therefore (5.12)3 is valid and Lemma 5.2 follows. \( \square \)
Thanks to the estimates in the previous lemma, we are now ready to pass to the limit $\epsilon \to 0$ and prove the existence of solution for our problem (5.1). The existence and uniqueness of solution of (5.1) is the object of the following lemma:

**Lemma 5.3.** The limit problem (5.1) possesses a unique solution $v \in L^\infty(\Omega)$ such that $v_y \in L^\infty(\Omega)$.

**Proof.** The uniqueness follows from the assumption (2.1). In fact, we suppose that there are two solutions $v_1, v_2$ to (5.1), then the difference $v_1 - v_2$ satisfies

$$
\frac{1}{2} \int_{y=0} (v_1 - v_2)^2 d\Gamma + \int_{\Omega} (p(v_1) - p(v_2)) (v_1 - v_2) = 0.
$$

Then (2.1) shows that $v_1 = v_2$ a.e.

We now show the existence of solution by passage to the limit $\epsilon \to 0$ in (5.3)-(5.4). To this end we introduce the following adjoint spaces for the operator $A$:

$$
(5.16) \quad D(A^*) = \{ v \in L^2(\Omega) \mid v_y \in L^2(\Omega), \nu(\cdot, 0) = 0 \}, \quad D(A^*_1) = D(A^*) \cap C^1(\Omega).
$$

We write (5.4a) with $\varphi \in D(A^*_1) \subset V$, and we obtain

$$
(5.17) \quad \epsilon \int_{\Omega} \nabla v^\epsilon \cdot \nabla \varphi + \int_{\Omega} \nu^\epsilon \varphi_y + \int_{\Omega} p(\nu^\epsilon) \varphi = \int_{\Omega} f \varphi, \forall \varphi \in D(A^*_1).
$$

From (5.12)_1 and (5.12)_2, we can extract a subsequence $\nu \to 0$, which for simplicity is still denoted by $\epsilon$, such that

$$
(5.18) \quad \begin{cases}
\epsilon^{\frac{1}{2}} \nabla \nu^\epsilon \to \chi \text{ weakly in } L^2(\Omega), \\
\nu^\epsilon \to v \text{ in } L^\infty(\Omega) \text{ weak-star}, \\
p(\nu^\epsilon) \to \beta \text{ in } L^\infty(\Omega) \text{ weak-star}.
\end{cases}
$$

Then, when $\epsilon \to 0$, (5.17) leads to

$$
(5.19) \quad \int_{\Omega} v\varphi_y + \int_{\Omega} \beta \varphi = \int_{\Omega} f \varphi, \forall \varphi \in D(A^*_1).
$$

Restricting $\varphi$ to the set of test functions in $C^0(\Omega)$, we conclude that

$$
(5.20) \quad -v_y + \beta = f,
$$

in the distribution sense on $\Omega$. Since $\beta, f \in L^\infty(\Omega)$, we see that $v_y \in L^\infty(\Omega)$. Therefore by a classical result concerning the derivatives of functions with values in a Banach space, see e.g. [T01] Chapter 3, Lemma 1.1, $v$ belongs to $C_y([0, L_2], L^\infty_y(0, L_1))$ and hence $v(\cdot, 0), v(\cdot, L_2)$ make sense and are in $L^\infty(0, L_1)$.

Now multiplying (5.20) with $\varphi \in D(A^*_1)$ and integrating by parts (which is legitimate), we obtain

$$
(5.21) \quad \int_{\Omega} v\varphi_y - \int_{y=L_2} v\varphi d\Gamma + \int_{\Omega} \beta \varphi = \int_{\Omega} f \varphi, \forall \varphi \in D(A^*_1).
$$

Comparing (5.19) and (5.21) yields

$$
\int_{y=L_2} v\varphi d\Gamma = 0, \forall \varphi \in D(A^*_1);
$$

this implies that $v(\cdot, L_2) = 0$. Hence we have proved that $v$ satisfies

$$
(5.22) \quad \begin{cases}
-v_y + \beta = f, \text{ in } \Omega, \\
v(x, L_2) = 0, \text{ for } x \in (0, L_1).
\end{cases}
$$
It is left to verify that \( \beta = p(v) \). To this end, we consider the term
\[
X_{\varepsilon}(u) = \langle A_{\varepsilon}(v^\varepsilon) - A_{\varepsilon}(u), v^\varepsilon - u \rangle \geq 0, \quad \text{for } u \in V.
\]
We have
\[
X_{\varepsilon}(u) = \langle A_{\varepsilon}(v^\varepsilon), v^\varepsilon - u \rangle - \langle A_{\varepsilon}(u), v^\varepsilon - u \rangle = \langle f, v^\varepsilon - u \rangle - \langle A_{\varepsilon}(u), v^\varepsilon - u \rangle.
\]
We observe that the second term can be rewritten as
\[
\langle A_{\varepsilon}(u), v^\varepsilon - u \rangle = \varepsilon \int_{\Omega} \nabla u \cdot \nabla (v^\varepsilon - u) + \int_{\Omega} u(v^\varepsilon - u_y) + \int_{y=0} u(v^\varepsilon - u) d\Gamma + \int_{\Omega} p(u)(v^\varepsilon - u)
\]
\[
= \varepsilon \int_{\Omega} \nabla u \cdot \nabla (v^\varepsilon - u) - \int_{\Omega} u_y(v^\varepsilon - u) + \int_{\Omega} p(u)(v^\varepsilon - u).
\]
When \( \varepsilon \to 0 \) we know that
\[
\begin{align*}
\varepsilon \nabla (v^\varepsilon - u) &\to 0 \text{ in } L^2(\Omega) \text{ weakly,} \\
v^\varepsilon &\rightharpoonup v \text{ in } L^2(\Omega) \text{ weakly.}
\end{align*}
\]
Hence when \( \varepsilon \to 0 \), we have
\[
\langle A_{\varepsilon}(u), v^\varepsilon - u \rangle \to - \int_{\Omega} u_y(v^\varepsilon - u) + \int_{\Omega} p(u)(v - u) = (-u_y + p(u), v - u).
\]
Thus, setting \( X(u) := \lim_{\varepsilon \to 0} X_{\varepsilon}(u) \), we easily see that
\[
X(u) = (f, v - u) - (-u_y + p(u), v - u)
\]
\[
= (-v_y + \beta, v - u) - (-u_y + p(u), v - u)
\]
\[
= -v_y - u_y + (\beta - p(u), v - u)
\]
\[
= \frac{1}{2} \int_{y=0} (v - u)^2 d\Gamma + \int_{\Omega} (\beta - p(u))(v - u) \geq 0.
\]
Since \( X_{\varepsilon}(u) \geq 0 \), we conclude that \( X(u) = 1/2 \int_{y=0} (v - u)^2 d\Gamma + \int_{\Omega} (\beta - p(u))(v - u) \geq 0, \forall u \in V \). By continuity and regularization in \( x \), we also have \( X(u) \geq 0 \), for all \( u \) such that \( u \in L^\infty(\Omega), u_y \in L^\infty(\Omega) \).
Taking \( u = v + \lambda w, \) for \( \lambda \in \mathbb{R} \) and \( w \in V \), we then have
\[
0 \leq X(v + \lambda w) = \frac{\lambda^2}{2} \int_{y=0} w^2 d\Gamma + \lambda \int_{\Omega} (\beta - p(u))w.
\]
We chose \( \lambda > 0 \); by dividing the above inequality for \( \lambda \) and letting \( \lambda \to 0 \), we obtain
\[
(\beta - p(u), w) = \int_{\Omega} (\beta - p(u))w \geq 0.
\]
Then replacing \( w \) by \(-w\), we also have
\[
(\beta - p(u), w) \leq 0.
\]
Hence
\[
(\beta - p(u), w) = 0, \quad \forall w \in V.
\]
We thus conclude that \( \beta = p(u) \). Hence the existence of solution for the limit problem (5.1) is shown. The lemma follows. \( \square \)
Proof. We first start with the regularity in \( y \). Then the periodic extension \( \tilde{v} \) of \( v \) belongs to \( C^\infty(\bar{\Omega}) \) as well and it satisfies (5.23).

We now have

Proposition 5.1. We assume that \( \tilde{f} \) is \( L_1 \) periodic in \( x \) and belongs to \( C^\infty(\bar{\Omega}) \). Then the periodic extension \( \tilde{v} \) of \( v \) belongs to \( C^\infty(\bar{\Omega}) \) as well and it satisfies (5.23).

We next derive the regularity in \( x \) for \( v \). By taking the partial derivatives in \( x \), we find that

\[
\begin{aligned}
\frac{\partial^{m+1}\tilde{v}}{\partial y^{m+1}} & = \frac{\partial^mp(\tilde{v})}{\partial y^m} - \frac{\partial f}{\partial y^m},
\end{aligned}
\]

Furthermore, using the Faà di Bruno formula for \( \frac{\partial^mp(\tilde{v})}{\partial y^m} \), we find

\[
\frac{\partial^m p(\tilde{v})}{\partial y^m} = \sum_{k=1}^m p^{(k)}(\tilde{v}) \sum_{(\alpha_1, \ldots, \alpha_{m-k+1})} m!(\frac{\partial \tilde{v}}{\partial y^m})^{\alpha_1} \cdots (\frac{\partial \tilde{v}}{\partial y^m})^{\alpha_{m-k+1}} \alpha_1! \cdots \alpha_{m-k+1}!
\]

where the \( \alpha_i \) are as in (2.2c) with \( j \) replaced by \( m \). By the induction hypotheses we find that \( \frac{\partial^mp(\tilde{v})}{\partial y^m} \) belongs to \( L_\infty(\bar{\Omega}) \), then so does \( \frac{\partial^{m+1}\tilde{v}}{\partial y^{m+1}} \). We next derive the regularity in \( x \) by considering the equations for \( v^m = \frac{\partial^m \tilde{v}}{\partial x^m} \). By taking the partial derivatives in \( x \) of equation (5.23), we obtain the equations for \( v^m \)

\[
\begin{aligned}
-\tilde{v}_y + p'(\tilde{v})v^m &= \frac{\partial f}{\partial x^m} - q^{m-1}(v^{m-1}) \quad \text{in } \Omega, \\
v^m(x, 1) &= 0, \quad \text{for all } x \in \mathbb{R}.
\end{aligned}
\]

Here \( q^{m-1}(v^{m-1}) = q^{m-1}(v, v^1, \ldots, v^{m-1}) = \frac{\partial^m p(\tilde{v})}{\partial x^m} - p'(\tilde{v})v^m \) is a function in \( \tilde{v}, v^1, \ldots, v^{m-1} \). By the Faà di Bruno formula, we find that

\[
q^{m-1}(v^{m-1}) = \sum_{k=1}^m p^{(k)}(\tilde{v}) \sum_{(\alpha_1, \ldots, \alpha_{m-k+1})} m!(v^{m-1})^{\alpha_1} \cdots (v^{m-k+1})^{\alpha_{m-k+1}} \frac{\alpha_1! \cdots \alpha_{m-k+1}!}{\alpha_1! \cdots \alpha_{m-k+1}!} - p'(\tilde{v})v^m,
\]

where the \( \alpha_i \) are as in (2.2c) with \( j \) replaced by \( m \). Some of the explicit formulas for \( q^n(v^n) \), \( n = 0, 1, 2, 3 \) are given by

\[
\begin{aligned}
q^0(v^0) &= q^0(\tilde{v}) = 0; \\
q^1(v^1) &= p''(\tilde{v})v^1; \\
q^2(v^2) &= p'''(\tilde{v})(v^1)^3 + 3p''(\tilde{v})v^1v^2; \\
q^3(v^3) &= p''(\tilde{v})(v^1)^4 + 6p'''(\tilde{v})(v^1)^2v^2 + 3p''(\tilde{v})(v^2)^2 + 4p''''(\tilde{v})v^1v^3.
\end{aligned}
\]
We multiply the first equation of (5.25) by \( e^{-\int_0^y p'(\tilde{v}(z))dz} \) and by explicit calculations, we find that:

\[
- \frac{\partial}{\partial y} \left( v^m(\cdot, y)e^{-\int_0^y p'(\tilde{v}(z))dz} \right) - \int_0^y \left( \frac{\partial}{\partial x^m} \tilde{f}(\cdot, t) - q^{m-1}(u^{m-1}) \right) e^{-\int_0^y p'(\tilde{v}(z))dz} dt
\]

or, using \( v^m(\cdot, 0) = 0 \),

\[
v^m(\cdot, y)e^{-\int_0^y p'(\tilde{v}(z))dz} = - \int_0^y \left( \frac{\partial}{\partial x^m} \tilde{f}(\cdot, t) - q^{m-1}(u^{m-1}) \right) e^{-\int_0^y p'(\tilde{v}(z))dz} dt.
\]

Therefore

\[
(5.28) \quad v^m(\cdot, y) = - \int_0^y \left( \frac{\partial}{\partial x^m} \tilde{f}(\cdot, t) - q^{m-1}(u^{m-1}) \right) e^{-\int_0^y p'(\tilde{v}(z))dz} dt.
\]

We know that \( v^0 = \tilde{v} \in L^\infty(\tilde{\Omega}) \). By induction (5.28) implies that \( \partial^m \tilde{v}/\partial x^m = v^m \in L^\infty(\tilde{\Omega}) \) for all \( m \). Because \( \partial^m \tilde{v}/\partial x^m \in L^\infty(\tilde{\Omega}) \) and \( \partial^m \tilde{v}/\partial y^m = v^m \in L^\infty(\tilde{\Omega}) \) for all \( m \), by taking the partial derivatives with respect to \( x \) of the equations in (5.24), we find that all the crossed partial derivatives \( \partial^m \partial^m \tilde{v} \in L^\infty(\tilde{\Omega}) \) for all \( m, n \). This finishes the proof of the lemma.

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References


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Table 1: The approximation errors for $f = \sin(\pi x)$ on $\Omega = (0, 2) \times (0, 1)$, $\epsilon$ vs numerical integration and exact formula in the evaluation of the integrations involving $p(u)$ when $\lambda \approx 1$. 

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<th>N</th>
<th>M</th>
<th>$\epsilon$</th>
<th>$L^2$ error</th>
<th>$L^\infty$ error</th>
<th>$\lambda$</th>
<th>$L^2$ error</th>
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Table 2: The approximation errors for $f = \sin(\pi x)$ on $\Omega = (0, 2) \times (0, 1)$; $M$, $N$ vs numerical integration and exact formula in the evaluation of the integrations involving $p(u)$ when $\lambda \approx 1$. 

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The table shows the approximation errors for $f = \sin(\pi x)$ on $\Omega = (0, 2) \times (0, 1)$, comparing numerical integration and exact formula for different $M$ and $N$ values when $\lambda \approx 1$. The errors are presented in both $L^2$ and $L^\infty$ norms, along with the corresponding $\lambda$ values.
Figure 3. $M = N = 40$, $f = \epsilon^y(1 - e^{-x})(1 - e^{-(1-x)})$, $\epsilon = 10^{-3}$ with Dirichlet boundary condition $u = 0$ on $\partial \Omega$: (a) $p(u) = u$; (b) $p(u) = u^3$; (c) $p(u) = u^5$. 