In this work, we consider the stochastic version of the diffusion equations with polynomial reaction terms forced by a multiplicative white noise. We establish the existence and uniqueness of a maximal pathwise solution for a limited period of time. The proof relies on the Skorohod representation theorem, the Gyöngy-Krylov theorem and stopping time arguments.

Contents

1 Introduction 2

2 Deterministic Systems of Reaction Diffusion Equations with Polynomial Growth 3

3 Stochastic Systems of Reaction Diffusion Equations with Polynomial Growth 5
   3.1 Functions Spaces .................................................. 6
   3.2 Stochastic Framework ............................................. 6
   3.3 Definition of Solutions .......................................... 9
   3.4 Main Result ...................................................... 10
   3.5 Formal a Priori Estimates ....................................... 10

4 The Modified System with a Cut Off Function 13
   4.1 The Approximate Systems ....................................... 13
   4.2 Uniform Estimates for the Approximate Systems .......... 14
   4.3 Compactness Arguments ......................................... 17
   4.4 Passage to the Limit ............................................. 19
   4.5 Global pathwise solution for the modified problem ...... 25
   4.6 Compactness revisited ......................................... 27
   4.7 Regularity in time of solutions ................................. 28

5 Existence and Uniqueness of Solutions for the Original System 29
   5.1 Local Martingale Solutions .................................... 29
   5.2 Local Pathwise Solutions ...................................... 30
   5.3 Maximal Pathwise Solutions .................................. 30
1 Introduction

Stochastic reaction diffusion equations have vast applications in biology, ecology, neuroscience, nanobiotechnology, etc. In the deterministic context, they can model various phenomena such as the models of predators and preys (Lotka Volterra models) [Hor07, NY96] or describe the free propagation of particles/molecules from a transmitter to a receiver in molecular communication [MYCA14, TMF+12], or model the birth-death processes at population levels [PS12]. In the stochastic versions, they can e.g. model metabolic processes [KS06], birth-death processes and random movements at the organism levels [DSH+13, Mas88], etc. The stochastic diffusion systems with polynomial growth reaction terms, that we consider here, can be e.g. Lotka Volterra systems driven by multiplicative white noise. These problems have been widely studied, in the settings of predators and preys, see e.g. [Dim03, DS06] or stochastic systems of neurons, see e.g. [BBD13] and the references therein.

The global existence for the odd order polynomials with positive energy reaction terms has been studied in the deterministic case in [Tem97] and in the stochastic case with multiplicative noise in [Fla91]. However, for other polynomial types of reaction terms, the solutions of the systems of reaction diffusion equations may blow up at finite time, see e.g. [FG02]; even for a single equation in one dimension, solutions in general blow up at a finite time, see e.g. [Eva90] for a detailed proof of a finite blow up time for \( u_t - u_{xx} = u^2 \).

For further background concerning the mathematical theory for the deterministic system of diffusion equations with polynomial reaction terms, see [Tem01] and the references therein. In the theory of stochastic evolution equations, two notions of solutions are typically considered namely the pathwise (or strong) solutions and the martingale (or weak) solutions. In the former notion, the driving noise is given in advance while in the later the underlying stochastic basis is unknown and will be found as part of the solution. For more details about the two types of solutions, we refer the readers to [DPZ92a], [Fla08], [FG95], and [Øks03]. In the study of nonlinear evolutionary partial differential equations, when we obtain the \( L^p \) bounds on the time approximation solution, the approximating equation will provide us estimates on the time derivatives, the classical compactness results can be applied to imply the existence of strong convergence of some subsequence in some suitable space, see [Tem01]. However, that compactness result cannot extend to the stochastic case since the solutions are not differentiable. We will utilize a different compactness result based on fractional Sobolev spaces that allows us to treat stochastic equations in a way similar to the deterministic case; see [FG95], [Tem95]. Proofs of other compactness embedding theorems can be found in [Bil95], [CF88], [Ros96], and [Tem01].

In this work, we will use the same approach as in [LNT16], [FG95] and [DGHT11] to establish the existence of both martingale and pathwise solutions but our results will provide finite time existence only since this is already the case in the deterministic context. We first provide a formal a priori estimate for the original stochastic system assuming that the solutions are sufficiently regular and we obtain uniform bounds for both types of solutions up to a strictly positive stopping time. However, unlike in the deterministic setting, quantitative lower bounds on this stopping times are unavailable but we do show that the stopping times
are positive almost surely. The absence of lower bound on the stopping times leads to further difficulties later on when deriving the compactness result and passing to the limit since we are not able to show the positiveness of the stopping time of the limit problem. To overcome these difficulties, we introduce the modified system which truncates the nonlinear term. We then obtain the existence of global martingale solutions for this system by using the Skorohod theorem. The next step is to prove the existence of a pathwise solutions for the modified system by proving that a pathwise uniqueness can be achieved. Having established the existence of martingale solutions and the pathwise uniqueness, the existence of global pathwise solutions follows by using the Gyöngy-Krylov theorem which is the infinite dimensional version of the classical Yamada-Watanabe theorem. Consequently, we derive the existence of both local martingale and pathwise solutions for the original stochastic system by introducing an appropriate positive stopping time afterward.

This article is organized as follows. In Section 2, we briefly recall the deterministic setting for the system and give a formal local in time a priori estimate for the solutions. Section 3 contains the stochastic background needed throughout the article. We also make precise the definitions of the solutions we are seeking and we also impose the necessary conditions on the noise through σ in this section. The core part of the manuscript lies in Section 4. In this section, we first introduce the modified (truncated) system and the Galerkin approximation scheme for that system and obtain the uniform estimates for the approximate solutions. These estimates are used to establish the compactness result which is the key step to infer the existence of the global martingale solutions along with their new underlying stochastic basis by making use of the Skorokhod theorem. We also successfully achieve the pathwise uniqueness of solutions and so by an application of the Gyöngy-Krylov theorem we can infer the existence of the pathwise solutions to the modified system. Another task in Section 4 is to define an appropriate strictly positive stopping time to obtain the local existence of both martingale and pathwise solutions of the original system. The appendix contains all the existing results and technical tools used throughout the body of the paper.

2 Deterministic Systems of Reaction Diffusion Equations with Polynomial Growth

Let \( \mathcal{M} \subset \mathbb{R}^n \) be open and bounded with smooth boundary \( \partial \mathcal{M} \). We consider the following (deterministic) system for \( T > 0 \):

\[
\frac{\partial u}{\partial t} - \nu \Delta u + \Phi(u) = f \quad \text{in } \mathcal{M} \times (0, T),
\]

\( u = 0 \) \quad \text{on } \partial \mathcal{M} \times (0, T), \quad (2.1a)

\[ u(t = 0) = u_0 \quad \text{in } \mathcal{M}. \quad (2.1b) \]

Here \( \nu \) is a given positive constant, \( f := f(x, t), u_0 := u_0(x) \) are given functions with values in \( \mathbb{R}^m \), and \( \Phi \) is a polynomial of degree \( q = |q|_1 = q_1 + q_2 + \cdots + q_n \), where \( q = (q_1, q_2, \ldots, q_n) \), namely:

\[
\Phi(u) = (\Phi_1(u), \ldots, \Phi_m(u))^T, \quad \text{with } \Phi_i(u) = \sum_{q \in \mathbb{N}^n : |q|_1 \leq q} a_q(x)u^q,
\]

where \( u^q = u_1^{q_1}u_2^{q_2} \cdots u_n^{q_n} \) and all of the \( a_q(x) \) are smooth, bounded functions.
For the mathematical settings of this problem, we write $H = L^2(\mathcal{M})^m$, $V = H^1_0(\mathcal{M})^m$. The inner product and the norm on $H$ are denoted by $\langle \cdot , \cdot \rangle$ and $| \cdot |$, respectively, while on $V$, we will use $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$, corresponding to the usual $L^2$--inner product and norm of the gradients.

**Remark 2.1.** In fact, we could also assume that the $a_q$ depend on time, $a_q = a_q(x, t)$, and are sufficiently regular, bounded functions in both the variables $x$ and $t$. The subsequent considerations are still valid in such case.

**Remark 2.2.** By the Young inequality, we easily obtain

$$|\Phi_i(u)| \leq C(1 + \sum_{i=1}^m |u_i|^q), \quad (2.2)$$

for some absolute constant $C = C(q, a_q)$. Thus

$$|\Phi(u)| \leq C(1 + |u|^q), \quad (2.3)$$

where $C = C(q, a_q, m)$ and $|u|$ is understood componentwise:

$$|u| = (u_1, \ldots, u_m)^T = (|u_1|, \ldots, |u_m|)^T.$$

**Lemma 2.1 (Local a priori estimate).** Suppose that $f \in L^2(0, T; L^2(\mathcal{M})^m)$ and $u_0 \in H^1_0(\mathcal{M})^m$ are given and that $u$ is a sufficiently regular solution of (2.1). Then

$$u \text{ belongs to an a priori bounded set of } L^\infty(0, t_*; V) \cap L^2(0, t_*; D(-\Delta)), \quad (2.4)$$

where

$$t_* = \left( |\nabla u_0|_{L^2(\mathcal{M})} + 1 \right)^{1-q} \frac{1}{2^q C_1}. \quad (2.5)$$

Here and below, $C(\cdot)$ is a constant depending on its arguments and this constant may be different at each occurrence.

**Proof of Lemma 2.1:**

We multiply (2.1a) by $v = -\Delta u$ and use the Stokes formula, the Cauchy Schwarz inequality and the Sobolev embeddings; we obtain:

$$\frac{1}{2} \frac{d}{dt} |\nabla u|^2 + v |\Delta u|^2 = -\langle \Phi(u), \Delta u \rangle + \langle f, \Delta u \rangle = \int_{\mathcal{M}} f \cdot \Delta u \, dM - \int_{\mathcal{M}} \Phi(u) \cdot \Delta u \, dM \leq |\Phi(u)| |\Delta(u)| + |f||\Delta(u)| \leq \frac{v}{4} |\Delta(u)|^2 + C(v)|\Phi(u)|^2 + \frac{v}{4} |\Delta(u)|^2 + \frac{v}{4} |f|^2 \leq \frac{v}{2} |\Delta(u)|^2 + C(v)(1 + \|u\|_{L^2(\mathcal{M})}^2) + \frac{v}{4} |f|^2 \leq \frac{v}{2} |\Delta(u)|^2 + C(v, \mathcal{M}, q)(1 + \|u\|^{2q}) + \frac{v}{4} |f|^2. \quad (2.6)$$
Rearranging and multiplying by 2 the above expression, we obtain:
\[
\frac{d}{dt} |\nabla u|^2 + v |\Delta u|^2 \leq 2C(v, \mathcal{M}, q)|\nabla u|^{2q} + 2 \left( C(v) + \frac{4}{q} |f|^2 \right) \leq C_1 (|\nabla u|^2 + 1)^q, \tag{2.7}
\]
with \( C_1 = C_1(v, f, \mathcal{M}, q) \). Then, setting \( y = |\nabla u|^2 \), we find
\[
\frac{dy}{dt} \leq C_1 (1 + y)^q. \tag{2.8}
\]
By integration of (2.8), we find that
\[
1 + y(t) \leq 2(1 + y_0), \tag{2.9}
\]
as long as
\[
1 + y(t) \leq 1 + y_0 + C_1 2^q (1 + y_0)^q t \leq 2(1 + y_0),
\]
which is true for for \( 0 \leq t \leq t_* \), in which \( t_* \) is specified as in (2.5). That is
\[
0 \leq t \leq t_* < \frac{(1 + y_0)^{1-q}}{2^q C_1}. \tag{2.10}
\]
Then, for \( 0 \leq t \leq t_* \), we have \( 1 + |\nabla u|^2 \leq 2(1 + |\nabla u_0|^2) \) and combine with (2.7) yield
\[
\frac{d}{dt} |\nabla u|^2 + v |\Delta u|^2 \leq 2^q C_1 (1 + |\nabla u_0|^2)^q. \tag{2.11}
\]
Integrating (2.11) in time over \([0, t]\) for \( 0 \leq t \leq t_* \) and taking the supremum over \([0, t_*]\), we finally deduce that
\[
\sup_{0 \leq t \leq t_*} |\nabla u(t)|^2 + v \int_0^{t_*} |\Delta u(t)|^2 dt \leq 2^{q+1} C_1 (1 + |\nabla u_0|^2)^q. \tag{2.12}
\]
which concludes the proof of the lemma.

For the detailed proofs of the existence and uniqueness of solutions of the (deterministic) system (2.1), we refer the readers to e.g. [Tem97].

\[\square\]

3 Stochastic Systems of Reaction Diffusion Equations with Polynomial Growth

We now consider the stochastic version of (2.1), by introducing a forcing \( \sigma(u) dW \) as explained below:
\[
du - v \Delta u dt + \Phi(u) dt = f dt + \sigma(u) dW \quad \text{in } \mathcal{M} \times (0, T), \tag{3.1a}
\]
supplemented with initial and boundary conditions:
\[
\begin{align*}
\quad u(t = 0) &= u_0 \quad \text{in } \mathcal{M}, \tag{3.1b} \\
\quad u &= 0 \quad \text{on } \partial \mathcal{M} \times (0, T). \tag{3.1c}
\end{align*}
\]

We first introduce the function spaces and the stochastic framework before introducing the definitions of the martingale and pathwise solutions of (3.1).
3.1 Functions Spaces

We have denoted by $H := L^2(\mathcal{M})^m$, $V = H_0^1(\mathcal{M})^m$ and we will continue to use these spaces throughout this work; the typical inner products and norms on $H$ and $V$ are denoted by $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle$ and $|\cdot|$, $\|\cdot\|$ respectively.

We also consider fractional powers of the $-\Delta$ operator with the Dirichlet boundary conditions. By the classical theory, there exists an orthonormal basis $\{\psi_k\}_{k \geq 1}$ of $H$ with unbounded increasing sequence of eigenvalues $\{\lambda_k\}_{k \geq 1}$ such that

$$-\Delta \psi_k = \lambda_k \psi_k.$$  

We then define $D(-\Delta) = V \cap H^2(\mathcal{M})^m$ and for $\alpha \geq 0$, we introduce the space

$$D((\Delta)^\alpha) = \{u \in H : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < \infty\}.$$  

endowed with the Hilbertian norm

$$|u|_\alpha := |(\Delta)^\alpha u| = \left( \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 \right)^{1/2}.$$  

Here, $u = \sum_{k=1}^{\infty} u_k \psi_k$ where $\sum_{k=1}^{\infty} |u_k|^2 < \infty$.

For the Galerkin scheme below, we introduce the finite dimensional spaces $H_n = \text{span}\{\psi_1, \ldots, \psi_n\}$ and let $P_n, Q_n = I - P_n$ be the projection operators in $H$ onto $H_n$ and onto its orthogonal complement. We have the generalized and reverse Poincaré inequalities which hold for any $\alpha_1 < \alpha_2$:

$$|P_n u|_{\alpha_2} \leq \lambda_n^{\alpha_2 - \alpha_1} |P_n u|_{\alpha_1} \quad \text{and} \quad |Q_n u|_{\alpha_1} \leq \frac{1}{\lambda_n^{\alpha_2 - \alpha_1}} |Q_n u|_{\alpha_2}. \quad (3.2)$$

3.2 Stochastic Framework

We define a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W_k\}_{k \geq 1})$ that is a filtered probability space with expectation $\mathbb{E}$. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ is a complete right continuous filtration, $\{W_k\}_{k \geq 1}$ is a sequence of independent one-dimensional Brownian motions relative to $\mathcal{F}_t$. We may formally define $W := \sum_k W_ke_k$, which makes each $W$ a cylindrical Brownian motion evolving over a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_k\}_{k \geq 1}$.

We now define what exactly we mean by a solution to problem (3.1) following [GHZ09]. First, we recall the definition of a predictable stochastic process:

**Definition 3.1.** For a given stochastic basis $\mathcal{S}$, let $\phi = \Omega \times [0, \infty)$ and let $\mathcal{G}$ be the $\sigma$-algebra generated by the sets of the form

$$(s, t] \times \mathcal{G}, \quad 0 \leq s < t < \infty, \quad \mathcal{G} \in \mathcal{F}_s; \quad \{0\} \times \mathcal{G}, \quad \mathcal{G} \in \mathcal{F}_0. \quad (3.3)$$

An $X$-valued process $U$ is called *predictable* w.r.t. $\mathcal{G}$ if it is measurable from $(\phi, \mathcal{G})$ into $(X, \mathcal{B}(X))$ where $\mathcal{B}(X)$ is the family of Borel sets of $X$. 


We next give the definitions of local and global solutions of (3.1) for both martingale and pathwise solutions. Before that, we make some assumptions for the initial condition \( u_0 \), which may be random in general. For the case of martingale solutions, since the stochastic basis is unknown, we are only able to specify \( u_0 \) as an initial probability measure \( u_0 \) on \( V \) such that \( \int_V \| x \|^p d\mu_0(x) < \infty \) for some \( p \geq 2 \). For the case of pathwise solutions where the stochastic basis \( S \) is fixed in advance, we assume that \( u_0 \) is a \( V \) valued random variable relative to this basis such that:

\[
\mathbf{u}_0 \in L^2(\Omega; V), \quad \text{and } \mathcal{F}_0 \text{-measurable.} \tag{3.4}
\]

We next recall some basic definition and properties of spaces of Hilbert-Schmidt operators. To this end, we suppose that \( X \) is another separable Hilbert space with the associated norm and inner product given by \( \| \cdot \|_X \) and \( (\cdot, \cdot)_X \), respectively.

We denote by \( L^2(\Omega, X) := \{ R \in \mathcal{L}(\Omega, X) : \sum\limits_{k=1}^{\infty} |Re_k|^2_X < \infty \} \) the collection of Hilbert-Schmidt operators mapping from \( \Omega \) to \( X \). This space \( L^2(\Omega, X) \) is a Hilbert space endowed with the following inner product and norm

\[
\langle R, S \rangle_{L^2(\Omega, X)} = \sum_{k=1}^{\infty} \langle Re_k, Se_k \rangle_X \quad \text{and} \quad \| R \|^2_{L^2(\Omega, X)} = \sum_{k=1}^{\infty} |Re_k|^2_X.
\]

We also define another auxiliary space \( \mathcal{U}_0 \supset \mathcal{U} \) as

\[
\mathcal{U}_0 := \left\{ u = \sum_{k=1}^{\infty} a_k e_k \in \mathcal{U} : \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty \right\},
\]

which is equipped with the norm

\[
|u|_{\mathcal{U}_0}^2 := \sum_{k=1}^{\infty} \frac{a_k^2}{k^2}, \quad \text{for } u = \sum_{k=1}^{\infty} a_k e_k.
\]

Note that the embedding of \( \mathcal{U}_0 \supset \mathcal{U} \) is Hilbert-Schmidt.

**Assumptions on \( \sigma \):** We shall assume throughout this work that

\[
\sigma := \sigma(u, t, \omega) : H \times [0, T] \times \Omega \rightarrow L_2(\Omega, H) \tag{3.5}
\]

is \( \mathcal{B}(H \otimes [0, T] \otimes \mathcal{F}, \mathcal{B}(L_2(\Omega, H)) \) measurable and essentially bounded in time and \( L^2 \) in \( \Omega \), \( \{ \mathcal{F}_t \}_{t \geq 0} \) adapted and satisfying the following conditions:

\[
\| \sigma(u, t, \omega) \|^2_{L_2(\Omega, V)} \leq K_V (1 + \| u \|^2), \tag{3.6a}
\]

\[
\| \sigma(u, t, \omega) \|^2_{L_2(\Omega, H)} \leq K_H (1 + \| u \|^2), \tag{3.6b}
\]

\[
\| \sigma(u_1, t, \omega) - \sigma(u_2, t, \omega) \|^2_{L_2(\Omega, V)} \leq K_V (\| u_1 - u_2 \|^2), \tag{3.6c}
\]

\[
\| \sigma(u_1, t, \omega) - \sigma(u_2, t, \omega) \|^2_{L_2(\Omega, H)} \leq K_H (\| u_1 - u_2 \|^2) \tag{3.6d}
\]
For simplicity, we shall denote \( \sigma(u, t, \omega) = \sigma(u) \).

Finally, given an \( X \)-valued predictable process \( G \in L^2(\Omega; L^2_{loc}([0, \infty); L_2(\Omega, X))) \) one may define the (Itô) stochastic integral

\[
M_t \ := \int_0^t G dW. \tag{3.7}
\]

If we write \( G_k = G \cdot e_k \) then \eqref{3.7} can be represented as

\[
M_t = \int_0^t G dW = \sum_k \int_0^t G_k dW_k,
\]

and it is an element of \( \mathcal{M}_X^2 \), the space of all \( X \)-valued square integrable martingales. As such \( \{M_t\}_{t \geq 0} \) has many desirable properties. Most notably for the analysis here, the Burkholder-Davis-Gundy inequality holds which in the present context takes the form,

\[
\mathbb{E} \left( \sup_{t \in [0, T]} \left\| \int_0^t G dW \right\|_X \right) \leq C_1 \mathbb{E} \left( \int_0^T \| G \|_{L_2(\Omega, X)}^2 dt \right)^{\frac{r}{2}}, \tag{3.8}
\]

which is valid for \( r \geq 1 \). If we write \( G = \sum_{k=1}^\infty G \cdot e_k \), \eqref{3.8} can be rewritten as

\[
\mathbb{E} \left( \sup_{t \in [0, T]} \left\| \int_0^t \sum_{k=1}^K G_k dW_k \right\|_X \right) \leq C_1 \mathbb{E} \left( \int_0^T \sum_{k=1}^K \| G_k \|_X^2 dt \right)^{\frac{r}{2}}. \tag{3.9}
\]

Here \( C_1 \) is an absolute constant depending on \( r \). We shall also make use of a variation of inequality \eqref{3.8}, which applies to fractional derivatives of \( M_t \). For \( p \geq 2 \) and \( \alpha \in [0, 1/2) \) we have

\[
\mathbb{E} \left( \left| \int_0^T G dW \right|_{W^{\alpha, p}([0, T]; X)}^p \right) \leq c \mathbb{E} \left( \int_0^T \| G \|_{L_2(\Omega, X)}^p dt \right), \tag{3.10}
\]

which holds for all \( X \)-valued predictable \( G \in L^2(\Omega; L^p_{loc}([0, \infty); L_2(\Omega, X))) \), see e.g. [DPZ92a].

We can express \eqref{3.10} in a form similar to \eqref{3.9}, normally

\[
\mathbb{E} \left( \left| \sum_k \int_0^T G \cdot e_k dW_k \right|_{W^{\alpha, p}([0, T]; X)}^p \right) \leq c \mathbb{E} \left( \int_0^T \sum_k \| G \cdot e_k \|_X^p dt \right). \tag{3.11}
\]

**Remark 3.1 (Notation).** For \( \sigma(u) \in L_2([0, T]; L_2(\Omega, H)) \) and \( W = \sum_{k=1}^\infty e_k W_k; \sigma(u) \) is represented formally by:

\[
\sigma(u) dW = \sum_{k=1}^\infty \sigma(u) \cdot e_k dW_k = \sum_{k, \ell=1}^\infty \langle \sigma(u) \cdot e_k, \phi_\ell \rangle \phi_\ell dW_k = \sum_{k, \ell=1}^\infty \sigma^{k, \ell} \phi_\ell dW_k, \tag{3.12}
\]

where

\[
\sigma^{k, \ell} = \langle \sigma(u) \cdot e_k, \phi_\ell \rangle,
\]

which makes sense since \( \sigma(u) \cdot e_k \in H \) and \( \{\phi_\ell\} \) is a Hilbert basis of \( H \).
3.3 Definition of Solutions

**Definition 3.2** (Local and global martingale solutions). Suppose $\mu_0$ is the initial probability measure on $V$ such that $\int_V \|u\|^p d(\mu_0) < \infty$ for some $p \geq 2$. We say that a triple $\left(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \bar{\mathbb{P}}, \bar{W}\right)$ is a local Martingale solution of problem (3.1) if $\bar{\Omega} := \left(\Omega, \mathcal{F}, \{(\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W\right)$ with expectation $\mathbb{E}$ is a stochastic basis, $\bar{\tau}$ is a strictly positive stopping time (i.e. $\tau > 0$ almost surely) relative to $\bar{\mathcal{F}}_t$, and $\bar{u}(\cdot \wedge \tau)$ is an $\bar{\mathcal{F}}_t$-adapted process in $V$ such that

$$
\bar{u}(\cdot \wedge \tau) \in L^2(\Omega; C([0, T]; V)), \\
1_{t \leq \tau} \bar{u} \in L^2(\Omega; L^2_{loc}(0, \infty; D(A))) = L^2(\Omega; L^2(0, T; D(-\Delta))),
$$

(3.13)

the law of $\bar{u}(0)$ is $\mu_0$, i.e. $\mu_0(E) = \mathbb{P}(\bar{u}(0) \in E)$ for all Borel subsets $E$ of $V$ and $\bar{u}$ satisfies for every $t \geq 0.$

$$
\bar{u}(t \wedge \tau) + \int_0^{t \wedge \tau} (-\nu \Delta \bar{u} + \Phi(\bar{u})) ds = \bar{u}_0 + \int_0^{t \wedge \tau} \sigma(\bar{u})dW + \int_0^{t \wedge \tau} f ds.
$$

(3.14)

We say that the Martingale solution $\left(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \bar{\mathbb{P}}, \bar{W}\right)$ is global if $\bar{\tau} = \infty$ a.s.

**Definition 3.3** (Local, maximal and global pathwise solutions). Suppose that $\Omega = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W)$ is a fixed stochastic basis and $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}, V)$.

(i) A pair $(u, \tau)$ is a local pathwise solution to (3.1) if $\tau$ is a strictly positive stopping time, $u(\cdot \wedge \tau)$ is an $\mathcal{F}_t$-adapted process in $V$ (relative to the fixed basis $\Omega$) such that (3.13)–(3.14) hold.

(ii) Pathwise solutions of (3.1) are said to be unique up to a stopping time $\tau > 0$ if given any pair of pathwise solutions $(u_1, \tau)$ and $(u_2, \tau)$ which coincide at $t = 0$ on a subset $\Omega_0$ of $\Omega$:

$$
\Omega_0 := \{u_1(0) = u_2(0)\} \subset \Omega,
$$

(3.15)

then

$$
\mathbb{P}\left(1_{\Omega_0}(u_1(t \wedge \tau) - u_2(t \wedge \tau)) = 0, \forall t \geq 0\right) = 1.
$$

(3.16)

(iii) Suppose we have $\{\tau_n\}_{n \geq 1}$, a strictly increasing sequence of stopping times that converge to a stopping time $\xi$, and assume that $u$ is a predictable continuous $\mathcal{F}_t$-adapted process in $V$. We say that $(u, \xi) := (u, \xi, \{\tau_n\}_{n \geq 1})$ is a maximal pathwise solution if $(u, \tau_n)$ is a local pathwise solution for each $n$ and

$$
\sup_{t \in [0, \xi]} |u(t)|^2 + \int_0^\xi |\Delta u|^2 ds = \infty,
$$

(3.17)

a.s. on the set $\{\xi < \infty\}$. If furthermore

$$
\sup_{t \in [0, \tau_n]} |u(t)|^2 + \int_0^{\tau_n} |\Delta u|^2 ds = n,
$$

(3.18)

for almost every $\omega \in \{\xi < \infty\}$, then the sequence $\tau_n$ announces a finite blow-up time.

(iv) If $(u, \xi)$ is a maximal pathwise solution and $\xi = \infty$ almost surely, then we say that the solution is global.
3.4 Main Result

We now state the main results in this work:

Theorem 3.1. Consider the space dimension \( n = 2 \) or \( 3 \), and \( q \) arbitrary if \( n = 2 \) and \( q \leq 2 \) if \( n = 3 \). We are given a probability measure \( \mu_0 \) on \( V \) such that \( \int_V \|x\|^p \, d\mu_0(x) < \infty \) for some \( p \geq 2 \) and we assume that \( f \in L^\infty(0, T; H) \), and \( \sigma \) satisfies the conditions \((3.6)\). Then there exists a local martingale solution \((\tilde{S}, \tilde{u}, \tilde{\tau})\) of \((3.1)\).

Theorem 3.2. We make the same assumptions for \( q \) and \( n \) as in Theorem 3.1. We are given a stochastic basis \( \mathcal{S} = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, \mathbb{W}) \), \( u_0 \) satisfying \((3.4)\), \( f \in L^\infty(0, T; H) \), and \( \sigma \) satisfying the conditions \((3.6)\). Then there exists a local pathwise solution \((u, \tau)\) of \((3.1)\) relative to \( \mathcal{S} \).

The strategy to prove both of theorems is that we first consider a modified system with a cut off function operating on the nonlinear term so that we have the global existence of martingale solutions for the modified system using Galerkin approximation. We then return to the original system by introducing a stopping time which will be proven to be positive almost surely.

3.5 Formal a Priori Estimates

We give a formal a priori estimate for the solutions of \((3.1)\) by assuming all the solutions are smooth. Applying the Itô’s formula to \( |\nabla u|^2 \) in \((3.1)\) yields

\[
d \|u\|^2 + 2v |\Delta u|^2 \, dt = -2 \langle f, \Delta u \rangle dt + 2 \langle \Phi(u), \Delta u \rangle dt + \sum_{k=1}^\infty \| \sigma(u) \cdot e_k \|^2 \, dt - 2 \sum_{k=1}^\infty \langle \sigma(u) \cdot e_k, \Delta u \rangle dW_k. \tag{3.19}
\]

Let \( M > 1 \); we define the stopping time based on the nonlocal property of the problem, which is the first time of escaping the ball of radius \( M \) of the solution:

\[
\tau := \tau_M = \inf_{t \geq 0} \left\{ \sup_{0 \leq r \leq t} \|u(r)\|^2 > M \right\}.
\]

Integrating \((3.19)\) in time over \([0, r]\) for \( 0 \leq r \leq t \wedge \tau \leq T \) and taking the supremum in \( r \), we find:

\[
\sup_{0 \leq r \leq t \wedge \tau} \|u(r)\|^2 + 2v \int_0^{t \wedge \tau} |\Delta u|^2 \, dt \leq 2 \|u(0)\|^2 + 4 \int_0^{t \wedge \tau} \|f, \Delta u\| \, dt + 4 \int_0^{t \wedge \tau} \|\Phi(u), \Delta u\| \, dt + 2 \sum_{k=1}^\infty \| \sigma(u) \cdot e_k \|^2 \, dt + 4 \sup_{0 \leq r \leq t \wedge \tau} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma(u) \cdot e_k, \Delta u \rangle dW_k \right|.
\]

\[
:= \|u(0)\|^2 + J_1 + J_2 + J_3 + J_4.
\]
We easily find the bound for $J_1$ by simply using the Cauchy-Schwarz inequality

$$J_1 = 4 \int_0^{t^\wedge \tau} |\langle f, \Delta u \rangle| dt \leq \frac{8}{v} \int_0^{t^\wedge \tau} |f|^2 dt + \frac{v}{2} \int_0^{t^\wedge \tau} |\Delta u|^2 dt.$$  \hfill (3.20)

The estimate for $J_2$ can be obtained by using the Cauchy-Schwarz inequality and the Sobolev embeddings. Recall from Sobolev embeddings that $H^1(\Omega)^m \subset L^{2q}(\Omega)^m$ for any $q \geq 1$ if $n = 2$ and $H^1(\Omega)^m \subset L^6(\Omega)^m$ if $n = 3$. Hence

$$J_2 := 4 \int_0^{t^\wedge \tau} |\langle \Phi(u), \Delta u \rangle| ds \leq 4 \int_0^{t^\wedge \tau} |\Phi(u)|_H |\Delta u|_H ds$$

$$\leq \frac{4}{v} \int_0^{t^\wedge \tau} |\Phi(u)|^2_H ds + \frac{v}{2} \int_0^{t^\wedge \tau} |\Delta u|^2 ds$$

$$\leq C_0 \int_0^{t^\wedge \tau} (1 + \|u\|^{2q}_{L^{2q}(\Omega)^m}) ds + \frac{v}{2} \int_0^{t^\wedge \tau} |\Delta u|^2 ds$$

$$\leq C_0 \int_0^{t^\wedge \tau} (1 + \|u\|^{2q}) ds + \frac{v}{2} \int_0^{t^\wedge \tau} |\Delta u|^2 ds. \hfill (3.21)$$

The third line holds true due to (2.3).

By utilizing the assumption (3.6), the bound for $J_3$ is direct

$$J_3 = 2 \int_0^{t^\wedge \tau} \sum_{k=1}^{\infty} \|\sigma(u) \cdot e_k\|^2 ds = 2 \int_0^{t^\wedge \tau} \|\sigma(u\|_{L^{2q}(\Omega)^m}) ds \leq 2K_V \int_0^{t^\wedge \tau} (1 + \|u\|^2) ds. \hfill (3.22)$$

Combining (3.20) – (3.22) and after taking the mathematical expectation on both sides, we arrive at:

$$\mathbb{E} \left( \sup_{0 \leq r \leq t^\wedge \tau} \|u(r)\|^2 + v \int_0^{t^\wedge \tau} |\Delta u|^2 ds \right)$$

$$\leq 2\mathbb{E}\|u(0)\|^2 + \mathbb{E} \left( \frac{8}{v} \int_0^{t^\wedge \tau} |f|^2 ds + 2K_V \int_0^{t^\wedge \tau} (1 + \|u\|^2) ds + C_0 M^{2q-2} \int_0^{t^\wedge \tau} (1 + \|u\|^2) ds \right)$$

$$+ 4\mathbb{E} \left( \sup_{0 \leq r \leq t^\wedge \tau} \left| \int_0^{r} \sum_{k=1}^{\infty} \langle \sigma(u) \cdot e_k, \Delta u \rangle dW^k \right| \right). \hfill (3.23)$$

Since $\sigma(u) \cdot e_k \in V$, $\Delta u \in H$, we obtain by integration by parts

$$\langle \sigma(u) \cdot e_k, \Delta u \rangle = \int_{\Omega} \sigma(u) \cdot e_k \Delta u d\Omega = -\int_{\Omega} \nabla \sigma(u) \cdot e_k \nabla u d\Omega = -(\nabla \sigma(u), \nabla u). \hfill (3.24)$$
Therefore, the stochastic term is majored by using the BDG inequality (3.9) with \( r = 1 \), \( Ge_k = \langle \sigma(u) \cdot e_k, \Delta u \rangle \), the Cauchy Schwarz inequality and (3.24):

\[
2\mathbb{E} \left( \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma(u) \cdot e_k, \Delta u \rangle dW^k \right| \right)^\frac{1}{2} \leq 2C_1 \mathbb{E} \left( \int_0^s \sum_{k=1}^\infty \langle \sigma(u) \cdot e_k, \Delta u \rangle^2 dt \right)^\frac{1}{2}
\]

\[
\leq 2C_1 \mathbb{E} \left( \int_0^s \sum_{k=1}^\infty \langle \nabla \sigma(u) \cdot e_k, \nabla u \rangle^2 dt \right)^\frac{1}{2} \leq 2C_1 \mathbb{E} \left( \int_0^s \sum_{k=1}^\infty \| \nabla \sigma(u) \cdot e_k \|^2 |\nabla u|^2 dt \right)^\frac{1}{2}
\]

\[
\leq C \mathbb{E} \left( \sup_{0 \leq r \leq s} \| u(r) \|^2 \int_0^s \sum_{k=1}^\infty \| \sigma(u) \cdot e_k \|^2 dt \right)^\frac{1}{2} \leq \frac{1}{2} \mathbb{E} \sup_{0 \leq r \leq s} \| u(r) \|^2 + C \mathbb{E} \left( \int_0^s \sum_{k=1}^\infty \| \sigma(u) \cdot e_k \|^2 dt \right)
\]

\[
\leq \frac{1}{2} \left( \mathbb{E} \sup_{0 \leq r \leq s} \| u(r) \|^2 + C_2 \mathbb{E} \left( \int_0^s \| u \|^2 dt \right) \right). \tag{3.25}
\]

Rearranging all estimates from (3.23) to (3.25), and multiplying by 2, we see that

\[
\mathbb{E} \left( \sup_{0 \leq r \leq t \wedge \tau} \| u(r) \|^2 + v \int_0^{t \wedge \tau} |\Delta u|^2 ds \right) \leq \mathbb{E} \left[ 4 \| u_0 \|^2 + \frac{16}{v} \int_0^{t \wedge \tau} |f|^2 ds \right]
\]

\[
+ \mathbb{E} \left[ 4K_2 \int_0^{t \wedge \tau} (1 + \| u \|^2) ds + 2C_0 M^{2q-2} \int_0^{t \wedge \tau} (1 + \| u \|^2) ds + 2C_2 \int_0^{t \wedge \tau} (1 + \| u \|^2) ds \right]
\]

\[
\leq \mathcal{K} \mathbb{E} \left( \int_0^{t \wedge \tau} (|f|^2 + 1) ds + \| u_0 \|^2 \right) + \mathcal{K} \mathbb{E} \left( \int_0^{t \wedge \tau} \| u \|^2 ds \right). \tag{3.26}
\]

where \( \mathcal{K} = 2 + 4K_2 + 2C_0 M^{2q-2} + 2C_2 + \frac{16}{v} \). Letting \( \mathcal{K}_1 = \mathcal{K} \mathbb{E} \left( \int_0^{T} (|f|^2 + 1) ds + \| u_0 \|^2 \right) \), we now define

\[
Y(t \wedge \tau) = \mathbb{E} \left( \int_0^{t \wedge \tau} \sup_{0 \leq r \leq s} \| u(r) \|^2 ds \right).
\]

Then (3.26) gives

\[
Y'(t \wedge \tau) \leq \mathcal{K}_1 + \mathcal{K} Y(t \wedge \tau).
\]

And it is not difficult to obtain that

\[
Y(t \wedge \tau) \leq \frac{\mathcal{K}_1}{\mathcal{K}} \left( e^{\mathcal{K}(t \wedge \tau)} - 1 \right). \tag{3.27}
\]

Combining (3.26) and (3.27), we see that:

\[
\mathbb{E} \left( \sup_{0 \leq r \leq t \wedge \tau} \| u(r) \|^2 + v \int_0^{t \wedge \tau} |\Delta u|^2 dt \right) \leq \mathcal{K}_1 + \mathcal{K}_1 \frac{\mathcal{K}_1}{\mathcal{K}} e^{\mathcal{K}(t \wedge \tau)} = \mathcal{K}_1 + \mathcal{K}_1 e^{\mathcal{K}(t \wedge \tau)}. \tag{3.28}
\]

The right hand side of (3.28) is bounded by \( M \) if

\[
\mathcal{K}_1 + \mathcal{K}_1 e^{\mathcal{K}(t \wedge \tau)} \leq M.
\]
or

\[ t \leq t_M := \frac{1}{K} \log \frac{M - K_1}{K_1}. \]

\( M \) is chosen large enough such that \( M - K_1 > K_1 \) or \( M > 2K_1 \). Then for \( 0 < t < t_M \wedge \tau \), we obtain the a priori estimates (3.28) for the solutions in \( L^2(\Omega, L^2(0, t_M \wedge \tau, D(-\Delta)) \cap L^2(\Omega, L^\infty(0, t_M \wedge \tau, V)). \)

4 The Modified System with a Cut Off Function

We aim to study the martingale solutions of the following modified system:

\[
\begin{align*}
    &d u - \nu \Delta u dt + \theta(||u||)\Phi(u) dt = fdt + \sigma(u) dW, \text{ in } \mathcal{M} \times (0, T), \\
    &u = 0 \text{ on } \partial \mathcal{M} \times (0, T), \\
    &u(0) = 0 \text{ in } \mathcal{M},
\end{align*}
\]  

(4.1a)

(4.1b)

(4.1c)

where \( \theta : \mathbb{R} \to [0, 1] \) is a \( C^\infty \) cut-off function satisfying

\[
\theta(x) = \begin{cases} 
1 & \text{if } x \leq K, \\
0 & \text{if } x \geq 2K.
\end{cases}
\]  

(4.2)

where \( K \) is any positive constant and is independent of \( n \).

**Theorem 4.1** (Global existence of martingale solutions to the modified system). With the same assumptions as in Theorem 3.1, there exists a global martingale solution to (4.1).

**Theorem 4.2** (Global existence of pathwise solutions to the modified system). With the same assumptions as in Theorem 3.2, there exists a global pathwise solution to (4.1) relative to the given stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1}) \).

4.1 The Approximate Systems

Aiming to define a martingale solution, we choose a stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1}) \) and we choose \( u_0 \in L^2(\Omega, V) \) with distribution \( \mu_0 \). We consider the sequence of the approximation systems \( (u^n) \) relative to this basis and initial condition.

We approximate the solutions of (4.1) by the Galerkin procedure:

We look for \( u^n, n \geq 1 \) to be solution of the SODEs resulting from projection of (3.1a) on \( H_n \); that is \( u^n \) solves the following system:

\[
\begin{align*}
    &d u^n - \nu \Delta u^n dt = P_n f dt - P_n[\theta(||u^n||)\Phi(u^n)] dt + \sum_{k=1}^{\infty} P_n \sigma(u^n) \cdot e_k dW^k, \\
    &u^n(0) = P_n u_0.
\end{align*}
\]  

(4.3a)

(4.3b)

Here \( u^n \) is an adapted process in \( \mathcal{C}([0, T]; H_n) \sim \mathcal{C}([0, T]; \mathbb{R}^n) \). The existence and uniqueness of \( u^n \) on \( (0, T) \) for a given \( T > 0 \) follows from the SODEs theory due to the Lipschitzian properties of the drift and diffusion terms, see e.g. [DPZ92a, Eva13].
4.2 Uniform Estimates for the Approximate Systems

We first derive some estimates on $u^n$ independently of $n$.

**Lemma 4.1.** Under the same assumptions as in Theorem 3.1, we have

$$u^n \text{ belongs to a set of } L^2(\Omega, L^\infty(0, T; V)) \cap L^2(\Omega, L^2(0, T; D(-\Delta))).$$

(4.4)

bounded independently of $n$.

**Proof of Lemma 4.1:** The proof can be justified in the same manner as in the Section 3.5. Thus, by replacing $u$ by $u^n$ in (3.19), we easily obtain the following

$$\sup_{0 \leq r \leq s} \|u^n(r)\|^2 + 2v \int_0^s |\Delta u^n|^2 \, dt \leq 2\|u^n(0)\|^2 + 4 \int_0^s \langle P_n[\theta(\|u^n\|)\Phi(u^n)], \Delta u^n \rangle \, dt$$

$$+ 4 \int_0^s \{P_n \Delta u^n\} \, dt + 2 \int_0^s \|P_n \sigma(u^n)\|_{L^2(\Omega, V)}^2 \, dt + 4 \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k \geq 1} \langle P_n \sigma(u^n) \cdot e_k, \Delta u^n \rangle dW_k \right|.$$

$$= \|u^n(0)\|^2 + J_1 + J_2 + J_3 + J_4.$$ (4.5)

Notice that we can drop $P_n$ in $J_1$, $J_2$ and $J_4$ because $P_n$ is self-adjoint and $P_n \Delta u^n = \Delta u^n$. We readily obtain the following estimates; note that by using (4.2) the nonlinear term can be treated with no effort.

First,

$$|J_1| \leq 4C \int_0^s |\Delta u^n| \, dt \leq \frac{v}{2} \int_0^s |\Delta u^n|^2 \, dt + \frac{8C^2 T}{v}.$$ (4.6a)

Then we split $J_2$ to obtain

$$|J_2| \leq \frac{v}{2} \int_0^s |\Delta u^n|^2 \, dt + \frac{8}{v} \int_0^s |f|^2 \, dt.$$ (4.6b)

Using the hypotheses (3.6), it follows that

$$|J_3| \leq 2KV \int_0^s (1 + \|u^n(t)\|^2) \, dt.$$ (4.6c)

Combining (4.5) with (4.6a), (4.6b), (4.6c) and taking mathematical expectation on both sides, we find

$$E \left( \sup_{r \in [0, s]} \|u^n(r)\|^2 + v \int_0^s |\Delta u^n|^2 \, dt \right) \leq \|u^n(0)\|^2 + \int_0^s (1 + |f|^2 + \|u^n\|^2) \, dt$$

$$+ \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k \geq 1} \langle P_n \sigma(u^n) \cdot e_k, \Delta u^n \rangle dW_k \right|.$$ (4.7)

Here and below the notation $\leq$ means “up to a multiplicative constant”.

---

**Page 14 of 37**
We find that the stochastic term is estimated exactly in the same as in (3.25):

\[
E \left( \sup_{\tau \in [0,s]} \int_0^\tau \sum_{k=1}^\infty \langle \nabla P_n \sigma(u^n) \cdot e_k, \nabla u^n \rangle dW_k \right) \\
\leq C_1 E \left( \int_0^s \sum_{k=1}^\infty \langle \nabla P_n \sigma(u^n) \cdot e_k, \nabla u^n \rangle^2 dt \right)^{\frac{1}{2}} \\
\leq C_1 E \left( \int_0^s \sum_{k=1}^\infty \|P_n \sigma(u^n) \cdot e_k\|^2 dt \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} E \left( \sup_{\tau \in [0,s]} \|u^n(\tau)\|^2 \right) + E \left( \int_0^s (1 + \|u^n\|^2) dt \right). \tag{4.8}
\]

Rearranging (4.7) – (4.8) and observing that \(\|u^n(0)\| = \|P^n u_0\| \leq \|u_0\|\), we find

\[
E \left( \sup_{\tau \in [0,s]} \|u^n(\tau)\|^2 + v \int_0^s |\Delta u^n|^2 dt \right) \leq E \left( \int_0^s (1 + |f|^2 + \|u^n\|^2) dt + \|u_0\|^2 \right). \tag{4.9}
\]

In particular,

\[
E \left( \sup_{\tau \in [0,s]} \|u^n(\tau)\|^2 \right) \leq E \left( \int_0^T (1 + \|f\|^2 + \|u^n\|^2) dt + \|u_0\|^2 \right). \tag{4.10}
\]

By applying the classical Gronwall inequality to \(Y(s) = E \left( \sup_{\tau \in [0,s]} \|u^n(\tau)\|^2 \right)\), we obtain

\[
E \left( \sup_{\tau \in [0,s]} \|u^n(\tau)\|^2 \right) \leq E \left( \int_0^s (1 + |f|^2) dt + \|u_0\|^2 \right). \tag{4.11}
\]

From (4.9) and (4.10), it is easy to conclude that

\[
E \left( \sup_{\tau \in [0,s]} \|u^n(\tau)\|^2 + v \int_0^s |\Delta u^n|^2 dt \right) \leq E \left( \int_0^s (1 + |f|^2 + \|u^n\|^2) dt + \|u_0\|^2 \right). \tag{4.11}
\]

The lemma is hence proven.

It is necessary to extend the result of Lemma 4.1 as follows; we will see it is a key step to establish the compactness result which helps establish the existence of the martingale solutions.

**Lemma 4.2.** Assume that \(f \in L^\infty(0, T; H)\) and for some \(p > 2\) we additionally assume that \(E(\|u_0\|^p) < \infty\).

We then obtain the moment estimate of order \(p\)

\[
E \left( \sup_{0 \leq t \leq T} \|u^n(t)\|^p \right) \leq K, \tag{4.12}
\]

where \(K\) only depends on the data.
Proof of Lemma 4.2. Neglecting the positive term $v \int_0^s |\Delta u^n|^2 \, dt$ in (4.7), raising both sides of (4.7) to the power $p/2$, we deduce:

$$\sup_{r \in [0,s]} \| u^n(r) \|^p \leq \| u_0 \|^p + \int_0^s |f|^p + \| u^n \|^p \, dt + \left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k \geq 1} (\nabla P_n \sigma(u^n) \cdot e_k, \nabla u^n) \, dW_k \right| \right)^{p/2}.$$  

We take the expected values on both sides of the above inequality and find

$$E \left( \sup_{r \in [0,s]} \| u^n(r) \|^p \right) \leq E \left( \| u_0 \|^p + \int_0^s (|f|^p + \| u^n \|^p + 1) \, dt \right)$$

$$+ E \left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k \geq 1} (\nabla P_n \sigma(u^n) \cdot e_k, \nabla u^n) \, dW_k \right| \right)^{p/2} \quad (4.13)$$

The stochastic term is evaluated as before using the BDG inequality in (3.9) by writing $G \cdot e_k = \langle \sigma^n(u^n) \cdot e_k, \Delta u^n \rangle$, $X = \mathbb{R}$, and the Cauchy-Schwarz inequality; we find

$$E \left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k \geq 1} (\nabla P_n \sigma(u^n) e_k, \nabla u^n) \, dW_k \right| \right)^{p/2} \leq C_1 E \left( \int_0^s \sum_{k \geq 1} |(\nabla P_n \sigma(u^n) e_k, \nabla u^n)|^2 \, dt \right)^{p/2} \leq C_1 E \left( \sup_{r \in [0,s]} \| u^n \|^2 \int_0^s \sum_{k \geq 1} \| P_n \sigma(u^n) e_k \|^2 \, dt \right)^{p/2} \leq C_1 E \left( \sup_{r \in [0,s]} \| u^n \|^2 \int_0^s (1 + \| u^n \|^p \, dt \right)^{p/2} \leq \frac{1}{2} E \left( \sup_{r \in [0,s]} \| u^n \|^p \right) + C E \left( \int_0^s (1 + \| u^n \|^p \, dt \right). \quad (4.14)$$

We combine (4.13) and (4.14) to deduce

$$E \left( \sup_{r \in [0,s]} \| u^n(r) \|^p \right) \leq E \left( \| u_0 \|^p + \int_0^s (|f|^p + \| u^n \|^p + 1) \, dt \right)$$

$$\leq E \left( \| u_0 \|^p + \int_0^s (|f|^p + 1) \, dt \right) + E \left( \sup_{r \in [0,T]} \| u^n(r) \|^p \, dt \right). \quad (4.15)$$

Applying the deterministic Gronwall inequality to $\mathcal{Y}(s) = E \left( \sup_{r \in [0,s]} \| u^n(r) \|^p \right)$, we finally obtain

$$E \left( \sup_{r \in [0,s]} \| u^n(r) \|^p \right) \leq E \left( \| u_0 \|^p + \int_0^s (|f|^p + 1) \, dt \right) \leq E \left( \| u_0 \|^p + \int_0^T (|f|^p + 1) \, dt \right). \quad (4.16)$$

Hence we completed the proof of the Lemma 4.2. □
Lemma 4.3 (Estimates in Fractional Sobolev Space). Under the same assumptions as in Theorem 3.1, we consider the associated sequence of solutions $(u^n)_{n \geq 1}$ of the Galerkin system (4.3). For any $\alpha \in [0, \frac{1}{2})$, there exists a positive constant $\kappa = \kappa(\alpha)$ such that

$$
\mathbb{E} \left( \left| \int_0^t P_n \sigma(u^n)dW \right|_{W^{\alpha,p}(0,T;H)}^p \right) \leq \kappa,
$$

(4.17)

$$
\mathbb{E} \left( \left| u^n(t) - \int_0^t P_n \sigma(u^n)dW \right|_{W^{1,2}(0,T;H)}^2 \right) \leq \kappa.
$$

(4.18)

where $\kappa$ only depends on the initial datum and $p$.

Proof of Lemma 4.3: We first derive (4.17). For $\alpha < \frac{1}{2}$, by using (3.10) and the Poincaré inequality, we obtain:

$$
\mathbb{E} \left( \left| \int_0^t P_n \sigma(u^n)dW \right|_{W^{\alpha,p}(0,T;H)}^p \right) \leq C \mathbb{E} \int_0^T |P_n \sigma(u^n)|_{L^p(U^H)}^p dt 
$$

$$
\leq C \mathbb{E} \int_0^T (1 + |u^n|^p) dt \leq C \mathbb{E} \int_0^T (1 + \|u^n\|^p) dt \leq \kappa.
$$

We now show (4.18) by rewriting the Galerkin system in integral form and rearrange terms, so that

$$
\int_0^t \sigma^n(u^n)dW = u_0^n - \int_0^t \left[ v \Delta u^n + \theta(||u^n||) P_n \Phi(u^n) - P_n \mathbf{f} \right] dt.
$$

Taking the mathematical expectation on both sides of the above equation, we find

$$
\mathbb{E} \left| \int_0^t \sigma^n(u^n)dW \right|_{W^{1,2}(0,T;H)}^2 \leq C \mathbb{E} \left| u_0^n \right|^2 + C \int_0^T \left[ v |\Delta u^n|^2 + \theta(||u^n||) |\Phi(u^n)|^2 + |\mathbf{f}|^2 \right] ds
$$

$$
\leq C \mathbb{E} \left| u_0^n \right|^2 + C \int_0^T \left[ v |\Delta u^n|^2 + 1 + \|u^n\|^2 + |\mathbf{f}|^2 \right] ds
$$

$$
\leq \kappa.
$$

The last line holds true in virtue of the Lemmas 4.1, 4.2 and the Sobolev embeddings, completing the proof of Lemma 4.3.

4.3 Compactness Arguments

We consider the phase spaces

$$
\chi_u = L^2(0, T; V) \cap \mathcal{C}([0, T]; V'), \chi_W = \mathcal{C}([0, T]; W_0), \text{ and } \chi = \chi_u \times \chi_W.
$$

(4.19)
We then define the probability measures
\[
\mu^n_u(\cdot) = \mathbb{P}(u^n \in \cdot) \in \text{Pr}(\chi_u), \quad \mu^n_w(\cdot) = \mathbb{P}(W \in \cdot) \in \text{Pr}(\chi_w).
\] (4.20)

Here \( \text{Pr}(\chi) \) is the set of all probability measures on \((\chi, \mathcal{B}(\chi)) \) with \( \mathcal{B}(\chi) \) being the associated Borel \( \sigma \)-algebra of \( \chi \). This defines a sequence of probability measures
\[
\mu^n = \mu^n_u \times \mu^n_w.
\] (4.21)
on the phase space \( \chi \).

We are going to show the tightness of \( \mu^n \) on \( \chi \):

**Lemma 4.4.** The sequence \( \mu^n \) is tight over \( \chi \) and hence weakly compact over the phase space \( \chi \).

**Proof of Lemma 4.4:** By using the Lemma 6.1 in the Appendix, we find that
\[
L^2(0, T; D(-\Delta)) \cap W^{1/2,2}([0, T]; H) \subseteq L^2(0, T; V).
\]

For \( R > 0 \), we define the set
\[
B_R^1 := \{ u \in L^2(0, T; D(-\Delta)) \cap W^{1/2,2}([0, T]; H) : |u|^2_{L^2(0, T; D(-\Delta))} + |u|^2_{W^{1/2,2}(0, T; H)} \leq R^2 \},
\]
which is compact in \( L^2(0, T; V) \).

By using Lemma 4.3, the Chebyshev inequality, and the interpolation inequality, we obtain
\[
\mu^n_u(B_R^1)^c = \mathbb{P}\left(|u|^2_{L^2(0, T; D(-\Delta))} + |u|^2_{W^{1/2,2}(0, T; H)} > R^2\right)
\leq \mathbb{P}\left(|u|^2_{L^2(0, T; D(-\Delta))} > \frac{R^2}{2}\right) + \mathbb{P}\left(|u|^2_{W^{1/2,2}(0, T; H)} > \frac{R^2}{2}\right)
\leq \frac{2}{R^2} \mathbb{E}\left(|u|^2_{L^2(0, T; D(-\Delta))}\right) + \frac{2}{R^2} \mathbb{E}\left(|u|^2_{W^{1/2,2}(0, T; H)}\right)
\leq \frac{k}{R^2}.
\] (4.22)

For \( p > 2 \) we choose \( \alpha \) such that \( \alpha p > 1 \). We also infer further the compact embedding as in Lemma 6.2 with \( \mathcal{E}_0 = H, \mathcal{E} = V' \),
\[
W^{1,2}(0, T; H) \subseteq \mathcal{C}([0, T]; V'), \quad \text{and} \quad W^{\alpha, p}(0, T; H) \subseteq \mathcal{C}([0, T]; V').
\]

Let \( B_R^{2,1} \) and \( B_R^{2,2} \) be the balls of radius \( R \) in \( W^{1,2}(0, T; H) \) and \( W^{\alpha, p}(0, T; H) \), respectively. It follows that
\[
B_R^2 = B_R^{2,1} + B_R^{2,2}
\]
is compact in \( \mathcal{C}([0, T]; V') \).

Observe that
\[
\{(u^n) \subset B_R^2 \} \supset \left\{ u^n - \int_0^t \sum_k P_n \sigma(u^n) \cdot e_k dW_k \in B_R^{2,1} \right\} \cap \left\{ \int_0^t \sum_k P_n \sigma(u^n) \cdot e_k dW_k \in B_R^{2,2} \right\}.
\]
By using Chebyshev inequality and combining with Lemma 4.3, we easily infer that

\[ \mu^R_n((B^2_R)^C) \leq \mathbb{P}\left( \left| u^n - \int_0^t \sum_{k=1}^{\infty} P_n \sigma (u^n) \cdot e_k dW_k \right|^2_{W^{1,2}(0,T;H)} \geq R^2 \right) + \mathbb{P}\left( \left| \int_0^t \sum_{k=1}^{\infty} P_n \sigma (u^n) \cdot e_k dW_k \right|^p_{W^{q,p}(0,T;H)} \geq R^p \right) \leq \frac{\kappa}{R^2}, \]

where \( \kappa \) is a generic constant independent of \( n \).

In view of applying Proposition 6.1 from the Appendix (and Definition 6.1), we take \( A^\varepsilon_u = B^1_R \) and for a given \( \varepsilon > 0 \), we define

\[ R = \left( C/(1 - \sqrt{1 - \varepsilon}) \right)^{\frac{1}{2}}. \]

Then by using this \( R \) in (4.22) and straightforward calculations in the resulting relation, we find

\[ \mu_n^u(A^\varepsilon_u) \geq \sqrt{1 - \varepsilon}. \]

The sequence \( \mu_n^W = \mu_W \) is tight by Proposition 6.1 since it is constant and hence is weakly compact. Thus there exists a compact set \( A^\varepsilon_W \subset W \) such that

\[ \mu_n^W(A^\varepsilon_W) \geq 1 - \varepsilon \geq \sqrt{1 - \varepsilon}, \]

for \( \tilde{\varepsilon} = 1 - \sqrt{1 - \varepsilon} \).

Thus setting \( A^\varepsilon = A^\varepsilon_u \times A^\varepsilon_W \), we conclude that

\[ \mu_n^A(A^\varepsilon) = \mu(A^\varepsilon_u)\cdot \mu(A^\varepsilon_W) \geq 1 - \varepsilon. \]

This proves that the sequence \( \mu_n^A \) is tight by Proposition 6.1.

\subsection*{4.4 Passage to the Limit}

From the tightness property and Prokhorov’s Theorem, there exists a subsequence \( n_j \) such that \( \mu_n^{n_j} \to \mu \) weakly where \( \mu \) is a probability measure on \( \mathcal{X} \). We associate those distributions to the approximate solution of the Galerkin scheme by stating the following proposition:

\begin{proposition}
Given a stochastic basis \( \mathcal{S} \), suppose that \( u_0 \) is associated with a probability measure \( \mu_0 \) on \( V \) such that \( \int_V \|x\|^p d\mu_0(x) < \infty \) for \( p \geq 2 \) and let \( u^n \) be the sequence defined above or a similar sequence. Then there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) with the associated expectation denoted by \( \tilde{\mathbb{E}} \), a subsequence \( n_j \to \infty \) and a sequence of \( \chi \)-valued random variables \((\tilde{u}^{n_j}, \tilde{W}^{n_j})\) such that

(i) The distribution of \((\tilde{u}^{n_j}, \tilde{W}^{n_j})\) is \( \mu^{n_j} \).
\end{proposition}
(ii) The distribution of $(\tilde{u}, \tilde{W})$ is $\mu$.

(iii) $(\tilde{u}^n_j, \tilde{W}^n_j)$ converges almost surely in the topology of $\tilde{\chi}$ to an element $(\tilde{u}, \tilde{W})$ i.e.

$$\tilde{u}^n_j \to \tilde{u}\text{ in } L^2(0, T; V) \cap C([0, T]; V') \text{ a.s.,} \quad (4.25a)$$

$$\tilde{W}^n_j \to \tilde{W}\text{ in } C([0, T]; \mathcal{U}_0) \text{ a.s..} \quad (4.25b)$$

(iv) Let $\tilde{F}^n_j := \sigma(\tilde{W}^n_j(s), \tilde{u}^n_j(s), s \leq t)$ which is the union of $\sigma$-algebras generated by a random variable $(\tilde{u}^n_j, \tilde{W}^n_j)$, then each $\tilde{W}^n_j$ is a cylindrical Wiener process with respect to the filtration $\tilde{F}^n_j$.

(v) Each pair $(\tilde{u}^n_j, \tilde{W}^n_j)$ satisfies $\tilde{\mathbb{P}}$-a.s.

$$\begin{align*}
    d\tilde{u}^n_j - v\Delta \tilde{u}^n_j + \theta(\|\tilde{u}^n_j\|)P_{n_j} \Phi(\tilde{u}^n_j) dt &= P_{n_j} f dt + \sum_{k=1}^{\infty} P_{n_j} \sigma(\tilde{u}^n_j) \cdot e_k d\tilde{W}^n_j, \\
    \tilde{u}^n_j &= P_{n_j} u_0.
\end{align*} \quad (4.25c)$$

Proof: We have shown in Lemma 4.4 that the sequence of measures $(\mu^n)_{n \geq 1}$ associated with the approximation scheme $(u^n, W)$ are weakly compact on $\Omega$. Thus, by a direct application of the Skorohod embedding theorem, the proofs of (i),(ii) and (iii) are granted see e.g. [DPZ92b]. In order to prove (iv), it suffices to show the followings, see [PR07]

(iii)$_A$: $\tilde{W}^n_j(t)$ is measurable with respect to $\tilde{F}^n_j$,

(iii)$_B$: $\tilde{W}^n_j(t) - \tilde{W}^n_j(s)$ is independent of $\tilde{F}^n_j$, $s \leq t$.

Proof of (iii)$_A$ and (iii)$_B$: The proofs for both of them are straightforward by using the definition of $\tilde{F}^n_j$ and note that $\sigma(\tilde{W}^n_j(t_1), \tilde{W}^n_j(t_2), ..., \tilde{W}^n_j(t_n)) = \sigma(\tilde{W}^n_j(t_1), \tilde{W}^n_j(t_2) - \tilde{W}^n_j(t_1), ..., \tilde{W}^n_j(t_n) - \tilde{W}^n_j(t_{n-1}))$.

In order to show (iv), we refer the readers to the technique of modification as in [Ben95].

We now show that $(\tilde{S}, \tilde{u})$ is indeed a global martingale solution of (4.3): We can see that due to (4.25c), all uniform estimates for $u^n$ are valid for $\tilde{u}^n_j$. Hence $\tilde{u}^n_j$ belongs to a bounded set of $L^2(\tilde{\Omega}, L^\infty(0, T; V)) \cap L^2(\tilde{\Omega}, L^2(0, T, D(-\Delta)))$. Thus there exists a subsequence still denoted by $n_j$ to save a notation and there exists $\tilde{u}$ in this intersection space such that:

$$\tilde{u}^n_j \to \tilde{u}\text{ weakly in } L^2(\tilde{\Omega}, L^2(0, T; D(-\Delta))), \quad (4.26)$$

and

$$\tilde{u}^n_j \to \tilde{u}\text{ weak-star in } L^2(\tilde{\Omega}, L^\infty(0, T; V)). \quad (4.27)$$
Integrating (4.25c) from 0 to \( t \) for \( 0 \leq t \leq T \), we arrive at

\[
\tilde{u}^{n,j}(t) - \int_0^t v \Delta \tilde{u}^{n,j} \, ds = \tilde{u}_0^{n,j} + \int_0^t P_{n,j} \, ds - \int_0^t P_{n,j} \left[ \theta(\|\tilde{u}^{n,j}\|) \Phi(\tilde{u}^{n,j}) \right] \, ds + \int_0^t \sum_{k=1}^{\infty} P_{n,j} \sigma(\tilde{u}^{n,j}) \cdot e_k \, d\tilde{W}_k^{n,j}.
\] (4.28)

Taking the inner product of (4.28) with \( \psi \in V \) yields the equation

\[
\langle \tilde{u}^{n,j}(t), \psi \rangle - \langle \int_0^t v \Delta \tilde{u}^{n,j} \, ds, \psi \rangle = \langle \tilde{u}_0^{n,j}, \psi \rangle + \langle \int_0^t P_{n,j} \, ds, \psi \rangle
\]

\[-\langle \int_0^t P_{n,j} \left[ \theta(\|\tilde{u}^{n,j}\|) \Phi(\tilde{u}^{n,j}) \right] \, ds, \psi \rangle + \langle \int_0^t \sum_{k=1}^{\infty} P_{n,j} \sigma(\tilde{u}^{n,j}) \cdot e_k \, d\tilde{W}_k^{n,j}, \psi \rangle.
\] (4.29)

Since \( \tilde{u}^{n,j} \to \tilde{u} \) in \( C([0,T];V') \) a.s., we can deduce the existence of a set \( \Omega_1 \subseteq \Omega \) such that \( \tilde{P}(\Omega_1) = 1 \) and on this set, the below convergence holds

\[
\lim_{j \to \infty} \langle \tilde{u}^{n,j}(0) - \tilde{u}(0), \psi \rangle_{L^2} = 0.
\] (4.30)

Set \( \tilde{\Omega} = \Omega \setminus \Omega_1 \) and we now show that the convergence of the other terms in (4.29) hold in \( L^2(\tilde{\Omega} \times [0,T]) \).

Due to the strong convergence (4.25a) and the estimates similar to (4.4) for the \( \tilde{u}^{n,j} \), using the Lebesgue Dominated Convergence Theorem, we see that \( \tilde{u}^{n,j} \) converges to \( \tilde{u} \) in \( L^2(\tilde{\Omega}, L^2(0,T,V)) \) strongly. Hence, after another extraction of subsequence, \( \tilde{u}^{n,j} \to \tilde{u} \) a.e. and \( \tilde{P} \)-a.s., that is there exists a subset \( \Omega_T^0 \subseteq \tilde{\Omega} \times [0,T] \) with full measure such that \( \forall (\omega,t) \in \Omega_T^0 \),

\[
\lim_{j \to \infty} \| \tilde{u}^{n,j} - \tilde{u} \| = 0.
\] (4.31)

From which we imply that \( \forall (\omega,t) \in \Omega_T^0 \)

\[
\lim_{j \to \infty} \langle \tilde{u}^{n,j}(t) - \tilde{u}(t), \psi \rangle_{L^2} = 0.
\] (4.32)

The convergence of the linear term is direct. Indeed, thanks to (4.26), there exists a set of full measure \( \Omega_T^1 \subseteq \tilde{\Omega} \times [0,T] \) with respect to \( d\tilde{P} \otimes dt \) and an extracted subsequence still denoted by \( \tilde{u}^{n,j} \) such that for all \( (\omega,t) \in \Omega_T^1 \),

\[
\left| \int_0^T (v \Delta \tilde{u}^{n,j}(s) - v \Delta \tilde{u}(s), \psi) ds \right| \leq \|\psi\| \left( \int_0^T \|\tilde{u}^{n,j}(s) - \tilde{u}(s)\|^2 ds \right)^{\frac{1}{2}} \to 0 \text{ as } j \to \infty.
\] (4.33)

Furthermore, in virtue of Lemma 4.1, we easily obtain

\[
\tilde{E} \int_0^T \left| \int_0^t v \langle \Delta \tilde{u}^{n,j}, \psi \rangle ds \right|^2 dt \leq \|\psi\|^2 \tilde{E} \left( \sup_{0 \leq t \leq T} \|\tilde{u}^{n,j}\|^2 \right).
\] (4.34)
From (4.33) and (4.34) and the Lebesgue Dominated Convergence Theorem, we conclude that
\[ \lim_{j \to \infty} \| v \int_0^t (\Delta \tilde{u}^{n_j}(t) - \Delta \tilde{u}, \psi)_{L^2} \|_{L^2(\Omega \times [0, T])} = 0. \] (4.35)

By extracting another subsequence, there exists a full set of measure \( \Omega^2_T \subset \tilde{\Omega} \times [0, T] \) w.r.t \( d\tilde{\mathbb{P}} \otimes dt \) such that for all \( (\omega, t) \in \Omega^2_T \),
\[ \lim_{j \to \infty} \int_0^t v(\Delta \tilde{u}^{n_j} - \Delta \tilde{u}, \psi)_{L^2} ds = 0. \] (4.36)

Since \( P_n \mathbf{f} \to \mathbf{f} \) in \( H \) for all \( \omega \in \tilde{\Omega} \) and \( |P_n \mathbf{f}|_{H} \leq |\mathbf{f}|_{H} \), we find by the Lebesgue Dominated Convergence theorem
\[ \int_0^t \langle P_n \mathbf{f}, \psi \rangle_{H} ds \to \int_0^t \langle \mathbf{f}, \psi \rangle_{H} ds \text{ in } L^2(\tilde{\Omega} \times [0, T]). \] (4.37)

Thus, we can deduce the existence of a set \( \Omega^3_T \subset \tilde{\Omega} \times [0, T] \) with full measure such that the above convergence holds pointwise for all \( (\omega, t) \in \Omega^3_T \).
We now pass to the limit in the nonlinear term; we write
\[
\left| \int_0^t \langle \theta(\|\tilde{u}^{n_j}\|) P_{n_j} \Phi(\tilde{u}^{n_j}) - \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle_{H} ds \right| 
\leq \left| \int_0^t \langle \theta(\|\tilde{u}^{n_j}\|) P_{n_j} \Phi(\tilde{u}^{n_j}) - \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle_{H} ds \right| 
\leq \left| \int_0^t \langle \theta(\|\tilde{u}^{n_j}\|) P_{n_j} \Phi(\tilde{u}^{n_j}) - \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle_{H} ds \right| 
\leq \left| \int_0^t \langle \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle_{H} ds \right| 
= I_1 + I_2. \] (4.38)

Each term is treated as follows:
For \( I_1 \): Due to (4.31), we see that \( \forall (\omega, t, x) \in \Omega^0_T \times \mathcal{M}, \)
\[ \lim_{j \to \infty} \theta(\|\tilde{u}^{n_j}\|) \Phi(\tilde{u}^{n_j}) = \theta(\|\tilde{u}\|) \Phi(\tilde{u}). \] (4.39)

Next due to (4.2), we derive the following bounds
\[ \int_0^t |\theta(\|\tilde{u}^{n_j}\|) \Phi(\tilde{u}^{n_j})|_{H} ds \leq \|\psi\| \int_0^T |\theta(\|\tilde{u}^{n_j}\|) \Phi(\tilde{u}^{n_j})| dt 
\leq \|\psi\| \int_0^T (1 + \mathcal{K}^P) \leq C. \] (4.40)

\[ \tilde{\mathbb{E}} \int_0^T \left| \int_0^t \langle \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle_{H} ds \right|^2 dt \leq \|\psi\|^2 \tilde{\mathbb{E}} \int_0^T (1 + \mathcal{K}^2P) \leq C. \] (4.41)

Combining (4.39), (4.40) and (4.41) and the Lebesgue Dominated Convergence Theorem, we see that
\[ \lim_{j \to \infty} \left\| \int_0^t \langle \theta(\|\tilde{u}^{n_j}\|) \Phi(\tilde{u}^{n_j}) - \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle_{H} ds \right\|_{L^2(\tilde{\Omega} \times [0, T])} = 0. \] (4.42)
By passing to a subsequence, we will see that there is a set of full measure \( \Omega_T^4 \subset \tilde{\Omega} \times [0, T] \) such that \( \forall (\omega, t) \in \Omega_T^4 \), we are able to obtain the following convergence

\[
\lim_{j \to \infty} \int_0^t \langle \theta(\|\tilde{u}^{n_j}\|) \Phi(\tilde{u}^{n_j}) - \theta(\|\tilde{u}\|) \Phi(\tilde{u}), \psi \rangle ds = 0. \tag{4.43}
\]

The term \( I_2 \) is evaluated in the similar way. More precisely, we infer from (3.2) and the uniform bound in (4.1) that:

\[
|\langle Q_{n_j} [\theta(\|\tilde{u}\|) \Phi(\tilde{u})], \psi \rangle| \leq \|\psi\|\|\theta(\|u\|)Q_{n_j} \Phi(\tilde{u})\| \leq \frac{1}{\lambda_j} \|\psi\|(1 + K^q) \to 0. \tag{4.44}
\]

The following estimates can be easily obtained by using (4.2) and (2.3):

- \[
\int_0^t \|\theta(\|u^{n_j}\|)Q_{n_j} \Phi(\tilde{u}), \psi\| ds \leq \|\psi\|(1 + \kappa^q), \tag{4.45}
\]

- \[
\int_0^T \left[ \int_0^t \|\theta(\|u^{n_j}\|) \Phi(\tilde{u}^{n_j}), \psi\| ds \right]^2 dt \\
\leq \mathbb{E} \int_0^T |\langle \theta(\|u^{n_j}\|) \Phi(\tilde{u}^{n_j}), \psi \rangle|^2 ds \leq \|\psi\|^2 \mathbb{E} \int_0^T (1 + \kappa^{2q}) dt \leq C. \tag{4.46}
\]

Combining (4.40) and (4.41) and by making another use of the Lebesgue Dominated Convergence Theorem, we conclude that

\[
\lim_{j \to \infty} \left\| \int_0^t \langle \theta(\|u^{n_j}\|)Q_{n_j} \Phi(\tilde{u}), \psi \rangle ds \right\|_{L^2(\Omega \times [0, T])} = 0. \tag{4.47}
\]

By extraction of another subsequence, there exists another full set \( \Omega_T^5 \) such that for all \( (\omega, t) \in \Omega_T^5 \), such that the convergence below holds for all \( (t, \omega) \in \Omega_T^5 \)

\[
\lim_{j \to \infty} \int_0^t \langle \theta(\|u^{n_j}\|)Q_{n_j} \Phi(\tilde{u}), \psi \rangle ds = 0. \tag{4.48}
\]

We address the stochastic term by using Lemma 6.4. From (4.25b), we know that \( \tilde{W}^{n_j} \to \tilde{W} \) in probability in \( \mathcal{C}(0, T; \mathcal{U}_0) \) and thus it suffices to show that \( P_{n_j} \sigma(\tilde{u}^{n_j}) \to \sigma(\tilde{u}) \) in \( L^2(0, T; L^2(\mathcal{U}, V)) \) except on a set of measure zero of \( \tilde{\Omega} \) and hence in probability. We utilize the Poincaré inequality, the hypothesis (3.6b), (3.2) and (4.31) and we estimate:

\[
\|P_{n_j} \sigma(\tilde{u}^{n_j}) - \sigma(\tilde{u})\|_{L^2(\mathcal{U}, V)}^2 \leq \|P_{n_j} \sigma(\tilde{u}^{n_j}) - P_{n_j} \sigma(\tilde{u})\|_{L^2(\mathcal{U}, V)}^2 + \|Q_{n_j} \sigma(\tilde{u})\|_{L^2(\mathcal{U}, V)}^2 \\
\leq \|\tilde{u}^{n_j} - \tilde{u}\|^2_V + \frac{1}{\lambda_{n_j}} (1 + \|\tilde{u}\|^2) \to 0 \text{ as } n_j \to \infty.
\]

Thus, we conclude that \( \|P_{n_j} \sigma(\tilde{u}^{n_j}) - \sigma(\tilde{u})\|_{L^2(\mathcal{U}, V)} \to 0, \forall (\omega, t) \in \Omega_T^0 \).

On the other hand, we observe that due to (3.6b) and (4.1),

\[
\mathbb{E} \left( \int_0^T P_{n_j} \|\sigma(\tilde{u}^{n_j})\|_{L^2(\mathcal{U}, V)}^2 dt \right) \leq C \mathbb{E} \left( \int_0^T (1 + \|\tilde{u}^{n_j}\|^2) \right) \leq C. \tag{4.49}
\]
With (4.44), (4.49) in hand and the Lebesgue Dominated Convergence Theorem, we infer that

\[ P_{n_j} \sigma(\tilde{u}^{n_j}) \to \sigma(\tilde{u}) \] in \( L^2(\tilde{\Omega}; L^2([0, T], L_2(\Omega, V))) \).

(4.50)

This implies that the following convergence holds almost surely and in particular, it holds in probability:

\[ P_{n_j} \sigma(\tilde{u}^{n_j}) \to \sigma(\tilde{u}) \] in \( L^2([0, T], L_2(\Omega, V)). \)

(4.51)

Combining with (4.25b), Lemma 6.4 is applied and we infer that

\[ \int_0^t P_{n_j} \sigma(\tilde{u}^{n_j}) d \tilde{W}^{n_j} \to \int_0^t \sigma(\tilde{u}) d \tilde{W} \] in \( L^2([0, T], V). \)

(4.52)

By making use of the Burkholder-Davis-Gundy inequality and the bound in (4.1), we can easily obtain the following estimate:

\[ \tilde{E}\left( \left\| \int_0^t P_{n_j} \sigma(\tilde{u}^{n_j}) d \tilde{W}^{n_j} \right\|_V^2 \right) \leq \tilde{E}\left( \sup_{0 \leq t \leq T} \left\| \int_0^t P_{n_j} \sigma(\tilde{u}^{n_j}) d \tilde{W}^{n_j} \right\|_V^2 \right) \]
\[ \leq C \tilde{E}\left( \int_0^T \| P_{n_j} \sigma(\tilde{u}^{n_j}) \|_{L_2(\Omega, V)}^2 dt \right) \leq C \tilde{E}\left( \int_0^T \| \sigma(\tilde{u}^{n_j}) \|_{L_2(\Omega, V)}^2 dt \right) \]
\[ \leq C \tilde{E}\left( \int_0^T (1 + \|\tilde{u}^{n_j}\|^2) dt \right) \leq C. \]

(4.53)

By utilizing the Lebesgue Dominated Convergence Theorem one more time, we obtain that the convergence in (4.50) holds further in \( L^2(\tilde{\Omega}; L^2([0, T], L_2(\Omega, V))). \) Hence, by the stochastic Fubini theorem, we can extract a subsequence and we find a set of full measure \( \tilde{\Omega}_T^6 \subset \tilde{\Omega} \times [0, T] \) such that the convergence of the stochastic term holds for all \((\omega, t) \in \tilde{\Omega}_T^6\).

Collecting all terms and setting \( \Omega_T = \bigcup_{i=0}^6 (\Omega_T^i)^C \) we imply that for all \((\omega, t) \notin \Omega_T\), the following convergence holds:

\[ \langle \tilde{u}(t), \psi \rangle = \lim_{j \to \infty} \langle \tilde{u}_{n_j}(t), \psi \rangle = \lim_{j \to \infty} \langle u_{n_j}(0), \psi \rangle - \lim_{j \to \infty} \left( \int_0^t v \Delta \tilde{u}_{n_j} ds, \psi \right) + \lim_{j \to \infty} \left( \int_0^t P_{n_j} f ds, \psi \right) \]
\[ - \lim_{j \to \infty} \left( \int_0^t P_{n_j} \theta(\|\tilde{u}^{n_j}\|) \Phi(\tilde{u}^{n_j}) ds, \psi \right) + \lim_{j \to \infty} \left( \int_0^t \sum_{k=1}^\infty P_{n_j} \sigma(\tilde{u}^{n_j}) \cdot e_k d \tilde{W}_k^{n_j}, \psi \right) \]
\[ = \langle u_0, \psi \rangle - \left( \int_0^t v \Delta \tilde{u} ds, \psi \right) - \left( \int_0^t f ds, \psi \right) - \left( \int_0^t \theta(\|\tilde{u}\|) \Phi(\tilde{u}) ds, \psi \right) \]
\[ + \left( \int_0^t \sum_{k=1}^\infty \sigma(\tilde{u}) \cdot e_k d \tilde{W}_k, \psi \right). \]

(4.54)

Since (4.54) holds for all \( \psi \in V \), by the density argument, this also holds true for \( \psi \in H \). We then conclude the proof of the proposition.
4.5 Global pathwise solution for the modified problem

We now prove that the global martingale solution for the system is pathwise unique.

**Proposition 4.2.** Suppose that \( u_1 \) and \( u_2 \) are two global martingale solutions of (4.1) relative to the same stochastic basis \( \mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W) \). We define \( \Omega_0 := \{ u_1(0) = u_2(0) \} \). Then we have uniqueness of pathwise solution that is \( u_1 \) and \( u_2 \) are indistinguishable on \( \Omega_0 \) in the sense that

\[
\mathbb{P} ( \mathbb{1}_{\Omega_0} (u_1(t) = u_2(t)), \forall t \geq 0 ) = 1.
\]

**Proof.** Let \( u = u_1 - u_2 \) and substitute \( u_1 \) and \( u_2 \) into (4.1), take the difference between these equations. We arrive at the following equation

\[
du - v \Delta u dt + [\theta(||u_1||)\Phi(u_1) - \theta(||u_2||)\Phi(u_2)]dt = \sum_{k=1}^{\infty} \sigma(u_1) \cdot e_k dW^k - \sum_{k=1}^{\infty} \sigma(u_2) \cdot e_k dW^k. \tag{4.55}
\]

By applying the Itô’s formula to \( \Psi(u) = ||u||^2 \), we derive an evolution equation for the V norm of this difference.

\[
d||u||^2 + 2v ||\Delta u||^2 dt = 2[\theta(||u_2||)\Phi(u_2) - \theta(||u_1||)\Phi(u_1)], \Delta u]dt
\]

\[
+ \sum_{k=1}^{\infty} ||\sigma(u_1)e_k - \sigma(u_2)|| e_k ||^2 + 2 \sum_{k=1}^{\infty} \{\sigma(u_1) \cdot e_k - \sigma(u_2) \cdot e_k, \Delta u\} dW_k. \tag{4.56}
\]

We integrate (4.56) in time over \([0, t], 0 \leq t \leq T\), multiply the resulting expression by \( \mathbb{1}_{\Omega_0} \) and take the expected values to obtain

\[
\mathbb{E} \mathbb{1}_{\Omega_0} \left( \sup_{s \in [0, t]} ||u(s)||^2 + 2v \int_0^t ||\Delta u||^2 ds \right) \leq \sum_{i=1}^{3} J_i, \tag{4.57}
\]

where the \( J_i \)'s are defined and estimated as follows:

\[
J_1 \leq 4\mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t ||(\theta(||u_2||)\Phi(u_2) - \theta(||u_1||)\Phi(u_1)), \Delta u|| dt \right)
\]

\[
\leq 4\mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t ||(\theta(||u_2||)\Phi(u_2) - \Phi(u_1)) + \theta(||u_2||) - \theta(||u_1||)\Phi(u_1)), \Delta u|| dt \right)
\]

\[
= J_1^1 + J_1^2. \tag{4.58}
\]

We estimate \( J_1^1 \) using the Cauchy Schwarz inequality and the Sobolev embeddings theorem:

\[
J_1^1 \leq \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t ||\Phi(u_2) - \Phi(u_1)|| ||\Delta u|| dt \right)
\]

\[
\leq \mathbb{E} \mathbb{1}_{\Omega_0} \left( C \int_0^t ||\Phi(u_2) - \Phi(u_1)||^2 + \frac{v}{6} ||\Delta u||^2 dt \right)
\]

\[
\leq \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t C ||u_1 - u_2||^2 (||u_1||^p + ||u_2||^p) \right) + \frac{v}{4} \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t ||\Delta u||^2 dt \right)
\]

\[
\leq \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t C(v, M, q) ||u||^2 (||u_1||^{2p} + ||u_2||^{2p}) \right) + \frac{v}{2} \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t ||\Delta u||^2 dt \right). \tag{4.59}
\]
We treat $J_1^2$ by making use of the Lipschitz property of $\theta$ and the bound (2.2) as follows:

\[
J_1^2 \leq \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t |\Phi(u_1)(\theta(\|u_2\|) - \theta(\|u_1\|))\Delta u| dt \right) \\
\leq \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t C\|u\|^2(1 + \|u_1\|^2) dt \right) + \frac{\nu}{2} \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t |\Delta(u)|^2 dt \right). \tag{4.60}
\]

The next term is bounded by simply using the hypothesis (3.6),

\[
J_2 = 2\mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t \sum_{k=1}^{\infty} \mathbb{1}_{\Omega_0}[\sigma(u_1) \cdot e_k - \sigma(u_2) \cdot e_k]^2 dt \right) \leq C \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t \|u\|^2 dt \right). \tag{4.61}
\]

The last stochastic term is evaluated by employing integration by part, the classical BDG inequality and the hypothesis (3.6c),

\[
J_3 = 2\mathbb{E} \mathbb{1}_{\Omega_0} \left( \sup_{s \in [0,t]} \left| \int_0^s \mathbb{1}_{\Omega_0}[\sigma(u_1) - \sigma(u_2)] \Delta u dW \right| \right) \\
\leq \frac{1}{2} \mathbb{E} \mathbb{1}_{\Omega_0} \left( \sup_{0 \leq s \leq t} \|u\|^2 ds \right) + C \mathbb{E} \mathbb{1}_{\Omega_0} \left( \int_0^t \|u\|^2 ds \right). \tag{4.62}
\]

Fix $\varepsilon > 0$, define the stopping times: $\tau_\varepsilon := \tau_{\varepsilon}^1 \wedge \tau_{\varepsilon}^2$, $\tau_{\varepsilon}^1 = \inf_{t \geq 0} \{2\|u_t\|^{2q} > \varepsilon \}$.

Rearranging (4.57)-(4.62), multiplying by 2 and replacing $T$ by $\tau_{\varepsilon} \wedge T$, we find

\[
\mathbb{E} \mathbb{1}_{\Omega_0} \left( \sup_{t \in [0,\tau_{\varepsilon}\wedge T]} \|u\|^2 + \nu \int_0^{\tau_{\varepsilon}\wedge T} |\Delta u|^2 dt \right) \\
\leq C(v, \mathcal{M}, q) \mathbb{E} \mathbb{1}_{\Omega_0} \int_0^{\tau_{\varepsilon}\wedge T} \left( 1 + 2\|u_1\|^{2p} + \|u_2\|^{2p} \right) \|u\|^2 dt.
\]

We now apply the stochastic Gronwall inequality Lemma 6.3 with $X = \sup_{0 \leq t \leq \tau_{\varepsilon} \wedge T} \|u\|^2$, $Y = |\Delta u|^2$, $R = 2\|u_1\|^{2p} + \|u_2\|^{2p} + 1$, $Z = 0$:

\[
\mathbb{E} \mathbb{1}_{\Omega_0} \left( \sup_{t \in [0,\tau_{\varepsilon}\wedge T]} \|u\|^2 + \nu \int_0^{\tau_{\varepsilon}\wedge T} |\Delta u|^2 dt \right) \leq \mathbb{E} \mathbb{1}_{\Omega_0} (\|u(0)\|^2) = 0. \tag{4.63}
\]

which implies that

\[
\mathbb{1}_{\Omega_0} \|u_1(t) - u_2(t)\|^2 = 0, \text{ a.s.} \tag{4.64}
\]

for $t \in [0, \tau_{\varepsilon} \wedge T]$.

From the definition of the stopping times, thanks to Chebyshev inequality and Lemma 4.2, we obtain that as $\varepsilon \to 0$, $\tau_{\varepsilon} \to \infty$. So for any $t \in [0, T]$, we have

\[
\mathbb{1}_{\Omega_0} \|u_1(t) - u_2(t)\|^2 = 0, \text{ a.s.} \tag{4.65}
\]

on a set of full measure which may depend on $t$. Taking the intersection of such sets corresponding to positive rational times, we infer:

\[
\mathbb{P}(\mathbb{1}_{\Omega_0} u_1(t) = u_2(t): t \in [0, T] \cap \mathbb{Q}) = 1. \tag{4.66}
\]

By the continuity in time of the solutions, we finally conclude that $u_1$ and $u_2$ are indistinguishable which proves the global pathwise uniqueness for the modified problem.
4.6 Compactness revisited

After establishing the existence of martingale solutions and pathwise uniqueness, we may apply the Gyöngy-Kylov theorem which is the infinite dimensional extension of the Yamada-Wantanabe Theorem to infer the existence of a global pathwise solution $u$ of the modified system. To do so, we return to the sequence $(u^n)$ of Galerkin solutions relative to the given stochastic basis $\mathcal{S}$. We argue in a similar manner to the compactness argument for global martingale solutions by considering the collections of joint distributions $v^{m,n} := \mu^m \times \mu^n$.

We consider the phase space $\chi^{n,W} = \chi_u \times \chi_W$ as defined in (4.19) and the laws $\mu^n_u, \mu^n_W = \mu^n_{W}, \mu^n$ as defined in (4.20) and (4.21). We next define

$$\chi^J = \chi_u \times \chi_u \times \chi_W.$$  \hspace{1cm} (4.67)

We then set

$$v^{m,n} := \mu^m \times \mu^n \times \mu_W.$$  \hspace{1cm} (4.68)

**Lemma 4.5.** The collection $\{v^{n,m}\}$ is tight and hence compact on $\chi^J$.

**Proof.** We follow the exact same proof as in Lemma 4.4. We take $B^1_R, B^2_R$ as in that proof and we can therefore choose $A_\epsilon, \tilde{A}_\epsilon$ compact in $\chi_u$ and $\chi_W$ respectively so that

$$\mu^n_u(A_\epsilon) \geq 1 - \frac{\epsilon}{3}, \mu^n_W(\tilde{A}_\epsilon) \geq 1 - \frac{\epsilon}{3}.$$  \hspace{1cm} (4.69)

We then take $\mathcal{K}_\epsilon = A_\epsilon \times A_\epsilon \times \tilde{A}_\epsilon$ which is compact in $\chi^J$. By (4.69), we see that

$$v^{m,n}(\mathcal{K}_\epsilon) \geq (1 - \frac{\epsilon}{3})^3 \geq 1 - \epsilon,$$

which holds for all $0 < \epsilon < 1$. The proof of the lemma is complete. \hfill $\square$

By the above lemma, $(v^{m,n})$ is tight and any subsequence of $\{v^{m,n}\}$ must also necessarily be tight.

By Prokhorov’s theorem, we may choose a subsequence $k'$ so that $\{v^{m_{k'},n_{k'}}\}$ converges weakly to an element $v'$. Then by an application of Skorohod’s theorem, we infer the existence of a new stochastic basis $\tilde{\mathcal{S}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ upon which is defined a sequence of random elements $(\tilde{u}^{m_{k'}}, \tilde{u}^{n_{k'}}, \tilde{W}^{n_{k'}})$ converging a.s. in $\chi^J$ to an element $(\tilde{u}, \tilde{u}, \tilde{W})$ in such a way that

$$\tilde{\mathbb{P}}((\tilde{u}^{m_{k'}}, \tilde{u}^{n_{k'}}, \tilde{W}^{n_{k'}}) \in \cdot) = v^{m_{k'},n_{k'}}(\cdot) \text{ and } \tilde{\mathbb{P}}((\tilde{u}, \tilde{u}, \tilde{W}) \in \cdot) = v'(\cdot).$$  \hspace{1cm} (4.70)

Note that in particular, $\mu^{m_{k'},n_{k'}}_u$ converges weakly to the measure $\mu_u$ defined by:

$$\mu_u(\cdot) = \tilde{\mathbb{P}}((\tilde{u}, \tilde{u}) \in \cdot).$$  \hspace{1cm} (4.71)

Let $\tilde{z}_{k'} = (\tilde{u}^{n_{k'}}, \tilde{W}^{n_{k'}}), \tilde{z}_{k'} = (\tilde{u}^{n_{k'}}, \tilde{W}^{n_{k'}})$ and $\tilde{z} = (\tilde{u}, \tilde{W}), \tilde{z} = (\tilde{u}, \tilde{W})$. We then infer that both $\tilde{u}$ and $\tilde{u}$ are two global martingale solutions of (3.1) over the same stochastic basis. One can prove that these
solutions agree with each other at time $t = 0$ a.s. and hence by uniqueness $\tilde{u} = \tilde{u}$ in $\chi_u$ a.s. In other words,

$$\mu_u(\{(x, y) \in \chi_u \times \chi_u : x = y\}) = \tilde{\mu}(\tilde{u} = \tilde{u} \text{ in } \chi_u) = 1.$$  \hfill (4.72)

With this conclusion, Proposition 6.3 implies that the original sequence $u^n$ defined on the initial probability space $(\Omega, \mathcal{F}, \mathbb{P})$ converges to an element $u$ in the topology of $\chi_u$, i.e.

$$u^n \rightarrow u \text{ a.s. in } L^2(0, T; V) \cap \mathcal{C}([0, T]; V').$$

By Proposition 4.1 we conclude that $u$ is a global pathwise solution of (3.1).

### 4.7 Regularity in time of solutions

The proofs for both Theorem 3.1 and Theorem 3.2 will be complete if we can upgrade the regularity in time for both martingale and pathwise solutions to match up with the regularity in (3.13). More precisely, we will show that $u \in \mathcal{C}([0, T]; V)$ a.s.

We are given $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W_k\}_{k \geq 1})$ and we let $Z$ be a solution of the stochastic system

$$dZ - \nu \Delta Z dt = f dt + \sigma(u) dW \text{ in } \mathcal{M} \times (0, T),$$ \hfill (4.73a)

supplemented with initial and boundary conditions:

$$Z(t = 0) = u_0 \text{ in } \mathcal{M}, \hfill (4.73b)$$

$$Z = 0 \text{ on } \partial \mathcal{M} \times (0, T), \hfill (4.73c)$$

where $u$ is a solution of (4.1). From classical SPDE theory, see e.g. [DPZ92a], we infer that

$$Z \in L^2(\Omega, L^2(0, T; D(-\Delta))) \cap L^2(\Omega, \mathcal{C}([0, T]; V')) \text{ provided } \sigma(u) \in \mathcal{L}_2(U, V). \hfill (4.74)$$

Let $\hat{u} = u - Z$ and subtract (4.73a) from (4.1a); we arrive at:

$$d\hat{u} - \nu \Delta \hat{u} dt + \theta(\|\hat{u} + Z\|) \Phi(\hat{u} + Z) dt = 0.$$ \hfill (4.75)

Since both $\Delta \hat{u}$ and $\theta(\|\hat{u} + Z\|) \Phi(\hat{u} + Z) \in L^2(\Omega, L^2(0, T; H))$, we infer that $\frac{d\hat{u}}{dt} \in L^2(\Omega, L^2(0, T; H))$. By the classical interpolation results, see e.g. [LM72], we find

$$\hat{u} \in L^2(\Omega, \mathcal{C}([0, T]; [D(A), H^1])) = L^2(\Omega, \mathcal{C}([0, T]; V)) \text{ a.s..} \hfill (4.76)$$

From (4.74) and (4.76), we conclude that

$$u = \hat{u} + Z \in L^2(\Omega, \mathcal{C}([0, T]; V)) \cap L^2(\Omega, L^2([0, T]; D(-\Delta))). \hfill (4.77)$$
5 Existence and Uniqueness of Solutions for the Original System

5.1 Local Martingale Solutions

Proposition 4.1 shows that \((\tilde{S}, \tilde{u})\) is a global Martingale solution to (4.1).

Now, we set

\[
\tau := \inf_{t \geq 0} \left\{ \sup_{0 \leq r \leq t} \|\tilde{u}(r)\|^2 > K \right\} \land T.
\]

(5.1)

where \(K\) is defined as in (4.2) and we specifically choose \(K = 1 + \|\tilde{u}_0\|^2\).

We can show in the following lemma that \(\tau\) is strictly positive and we observe that

\[
\int_0^{\tau \land T} \Phi(\tilde{u}(s))ds = \int_0^{\tau \land T} \Phi(\tilde{u}(s))ds.
\]

Thus \((\tilde{S}, \tilde{u})\) is a local martingale solution of (3.1). The proof of Theorem 3.1 is complete.

Lemma 5.1. The stopping time \(\tau\) in (5.1) is strictly positive.

Proof. Let \(\epsilon > 0\) be given. By the definition of \(\tau\), it is easy to see that:

\[
\{\tau < \epsilon\} \subseteq \left\{ \sup_{s \in [0, \tau \land \epsilon]} \|\tilde{u}(s)\|^2 - \|\tilde{u}_0\|^2 > 1 \right\}.
\]

From which and by Chebyshev’s inequality, we obtain:

\[
P(\tau = 0) = P(\cap_{\epsilon > 0}\{\tau < \epsilon\}) = \limsup_{\epsilon \to 0} P(\{\tau < \epsilon\}) \leq \limsup_{\epsilon \to 0} E \left( \sup_{s \in [0, \tau \land \epsilon]} \|\tilde{u}(s)\|^2 - \|\tilde{u}_0\|^2 \right).
\]

Thus, the desired result will be obtained once we can show that

\[
\limsup_{\epsilon \to 0} E \left( \sup_{s \in [0, \tau \land \epsilon]} \|\tilde{u}(s)\|^2 - \|\tilde{u}_0\|^2 \right) = 0. \tag{5.2}
\]

For that purpose, we replace \(u^n\) by \(\tilde{u}\) and \(s\) by \(\tau \land \epsilon\), we also drop \(P_n\) in (4.5), and we obtain:

\[
\sup_{0 \leq t \leq \tau \land \epsilon} \|\tilde{u}(t)\|^2 + 2\nu \int_0^{\tau \land \epsilon} |\Delta \tilde{u}|^2 dt \\
\leq 4 \int_0^{\tau \land \epsilon} |\theta(\|\tilde{u}\|)\Phi(\tilde{u}), \Delta \tilde{u}| dt + \\
+ 2 \int_0^{\tau \land \epsilon} \|\sigma(\tilde{u})\|_{L_2(\mu,V)}^2 dt + 4 \sup_{0 \leq t \leq \tau \land \epsilon} \int_0^t \sum_{k=1}^\infty (\sigma(\tilde{u}) \cdot e_k, \Delta \tilde{u})dW^k \bigg| + \|\tilde{u}_0\|^2 \\
= \|\tilde{u}_0\|^2 + J_1 + J_2 + J_3 + J_4. \tag{5.3}
\]
All the $J_i$’s terms on the right hand side of the above equation can be estimated in the similar way as we carried out the uniform estimates for $u^n$ and we arrive at the conclusion

$$\mathbb{E} \left( \sup_{t \in [0, \tau \wedge \epsilon]} \| \tilde{u} \|^2 + v \int_0^{\tau \wedge \epsilon} |\Delta \tilde{u}|^2 \, dt \right) \leq \mathbb{E} \left( \int_0^{\tau \wedge \epsilon} (1 + |f|^2) \, dt + \| \tilde{u}_0 \|^2 \right).$$

In particular, we obtain

$$\mathbb{E} \left( \sup_{t \in [0, \tau \wedge \epsilon]} \| \tilde{u} \|^2 - \| \tilde{u}_0 \|^2 \right) \leq \mathbb{E} \left( \int_0^{\tau \wedge \epsilon} (1 + |f|^2) \, dt \right). \tag{5.5}$$

Consequently,

$$\limsup_{\epsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, \tau \wedge \epsilon]} \| \tilde{u} \|^2 - \| \tilde{u}_0 \|^2 \right) \leq \limsup_{\epsilon \to 0} \mathbb{E} \left( \int_0^{\tau \wedge \epsilon} (1 + |f|^2) \, dt \right) \tag{5.6}$$

$$\leq \limsup_{\epsilon \to 0} \epsilon \mathbb{E} \left( 1 + |f|_{L^\infty(0,T;L^2(H))} \right) = 0.$$ 

Therefore, we completed the proof of the lemma. \hfill \Box

### 5.2 Local Pathwise Solutions

We let $\tau$ be as in (5.1), and use an identical argument to Section 4.6 to conclude that $(v, \tau)$ is a local pathwise solution of (3.1).

### 5.3 Maximal Pathwise Solutions

In this section, we aim to establish the existence of a maximal pathwise solution $(u, \xi)$ of the system (3.1).

Loosely speaking, if the pathwise solution exists up to the time $\tau$, one can uniquely extend this solution up to some stopping time $\tau_1 > \tau$. This will be precisely stated in the next lemma.

**Lemma 5.2.** Assume that $(u, \tau)$ is a local pathwise solution as established in the previous section, and that $\tau$ is finite almost surely. Then there exists a local pathwise solution $(u_\epsilon, \tau_1)$ such that $\tau_1 > \tau$ almost surely and such that $\mathbb{P}(u_\epsilon(t \wedge \tau) = u(t \wedge \tau) ; t \in [0, T]) = 1$.

**Proof.** We refer the readers to the work of Glatt-Holtz and Ziane [GHZ09, Lemma 4.1]. \hfill \Box

We also see that the local pathwise solution can be extended in time to be a maximal solution.

**Proposition 5.1.** Given $\mathcal{B} := \left( \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1} \right)$, then there exists a unique maximal solution $(u, \xi)$ relative to that stochastic basis and a sequence $\rho_R$ announcing $\xi$.
Proof. With the uniqueness already proved, we consider the set \( \mathcal{L} \) of all stopping times such that \( \tau \in \mathcal{L} \) if and only if there exist processes \( u \) s.t. \( (u, \tau) \) is a local pathwise solution. Clearly if two stopping times are in \( \mathcal{L} \), then so is their maximum and if \( \sigma \in \mathcal{L} \), thus \( \rho \land \sigma \in \mathcal{L} \) where \( \rho \) is any stopping time. Let \( \xi = \sup \mathcal{L} \) and choose an increasing sequence \( \tau_k \in \mathcal{L} \) such that \( \tau_k \to \xi \) a.s.

For each \( \tau_k \), denote by \( u_k \) the corresponding process that makes \( (u_k, \tau_k) \) a local pathwise solution. Let

\[
\Omega_{k,k'} = \{ u_k (t \land \tau_k \land \tau_{k'}) = u_{k'} (t \land \tau_k \land \tau_{k'}) : t \geq 0 \}. \tag{5.7}
\]

Then, by uniqueness, we see that \( \hat{\Omega} = \cap_{k,k'} \Omega_{k,k'} \) is a set of full measure. For \( \omega \) in this set and every \( t > 0 \), the sequence \( \{ u_k (t \land \tau_k) 1_{t < \xi} \} \) is Cauchy in \( H \). Let

\[
\tilde{u}(t) = \lim_{k \to \infty} u_k (t \land \tau_k) 1_{t < \xi} \quad \text{a.s..} \tag{5.8}
\]

Then by Lemma 4.1 and the Monotone Convergence Theorem, for any \( T > 0 \), we have

\[
E \left( \sup_{t \in [0,T]} \| \tilde{u} \|^2 + \int_0^{\xi \land T} |\Delta \tilde{u}|^2 dt \right) < \infty. \tag{5.9}
\]

We may then define \( u(t) \in H \) by:

\[
(u(t), v) + \int_0^{t \land \xi} (-v \Delta \tilde{u} + \Phi(\tilde{u}), v) = (u_0, v) + \int_0^{t \land \xi} (f, v) ds + \int_0^{t \land \xi} \sum_k (\sigma(\tilde{u}), v) dW^k. \tag{5.10}
\]

for any \( t > 0, v \in H \). Clearly for \( t < \xi(\omega) \), \( u(t, \omega) = \tilde{u}(t, \omega) \) and \( u \) is weakly continuous a.s. in \( H \). Thus, \( (u, \xi) \) is a local pathwise solution.

For \( R > 0 \), define the stopping time

\[
\rho_R := \inf_{t \geq 0} \left\{ \sup_{x \in [0,t]} \| u \|^2 + \int_0^t |\Delta u|^2 ds > R \right\} \land \xi. \tag{5.11}
\]

Then \( (u, \rho_R) \) is a local pathwise solution for any \( R > 0 \) and \( \{ \rho_R \}_{R \geq 0} \) announces \( \xi \) from Lemma 5.2.

\( \square \)

We have thus completed the proof of Theorem 3.2. We have proven Theorem 3.1 in Section 5 and we have now proven the two main results of this article.
6 Appendices

Appendix A

Suppose that $\mathcal{H}$ is a separable Hilbert space. Given $p \geq 2, \alpha \in (0, \frac{1}{2})$, we define the fractional derivative space $W^{\alpha,p}(0,T; \mathcal{H})$ as the Sobolev space of all $u \in L^p(0,T; \mathcal{H})$ such that

$$
\int_0^T \int_0^T \frac{|u(t) - u(s)|_\mathcal{H}^p}{|t-s|^{1+\alpha p}} dt ds < \infty,
$$

endowed with the norm

$$
|u|^p_{W^{\alpha,p}(0,T; \mathcal{H})} = \int_0^T |u(t)|_\mathcal{H}^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_\mathcal{H}^p}{|t-s|^{1+\alpha p}} dt ds.
$$

For the case $\alpha = 1$, we take $W^{1,p}(0,T; X) = \{ u \in L^p(0,T; X) : \frac{du}{dt} \in L^p(0,T; X) \}$ to be the classical Sobolev space with its usual norm.

We have applied the following lemmas, the proofs of which can be found in e.g. [Fla08] and [Tem95]:

**Lemma 6.1.** Let $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}_1$ be Banach spaces with the injections being continuous and $\mathcal{E}_0, \mathcal{E}_1$ reflexive. For $p \in (1, \infty), \alpha \in (0, 1)$, we have

$$
L^p(0,T; \mathcal{E}_0) \cap W^{\alpha,p}(0,T; \mathcal{E}_1) \subset L^p(0,T; \mathcal{E}).
$$

**Lemma 6.2.** If $\mathcal{E} \subseteq \tilde{\mathcal{E}}$ are Banach spaces and $p \in (1, \infty), \alpha \in (0, 1]$ are such that $\alpha p > 1$, then

$$
W^{\alpha,p}(0,T; \mathcal{E}) \subset C([0,T]; \tilde{\mathcal{E}}).
$$

We additionally often use the following stochastic version of the Gronwall lemma (see e.g. [GHZ09]):

**Lemma 6.3.** Fix $T > 0$ and assume that $X,Y,Z,R : \Omega \times [0,T) \to \mathbb{R}$ are non-negative stochastic processes. Let $\tau < T$ be a stopping time such that

$$
\mathbb{E}\left( \int_0^\tau (RX + Z) ds \right) < \infty \quad \text{and} \quad \int_0^\tau Rd s < \kappa, \quad \text{a.s.}
$$

Suppose that for all stopping times $0 \leq \tau_a \leq \tau_b \leq \tau$

$$
\mathbb{E}\left( \sup_{\tau \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds \right) \leq C_0 \mathbb{E}\left( X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \right).
$$

\[1\] $\|u\|_{W^{1,p}(0,T; X)} = \int_0^T |u(s)|_X^p ds + \int_0^T \left| \frac{du}{dt}(s) \right|_X^p ds$. Note that for $\alpha \in (0,1), W^{1,p}(0,T; X) \subset W^{\alpha,p}(0,T; X)$ and $\|u\|_{W^{\alpha,p}(0,T; X)} \leq C \|u\|_{W^{1,p}(0,T; X)}$. 

Page 32 of 37
where $C_0$ is independent of $\tau_a$ and $\tau_b$. Then
\[
\mathbb{E}\left( \sup_{t \in [0, \tau]} X + \int_0^\tau Y ds \right) \leq CE\left( X(0) + \int_0^\tau Z ds \right),
\]
where $C$ is a constant depending only on $C_0, T,$ and $\kappa$.

Finally, we require the Vitali convergence theorem (see e.g. [Fol99]):

**Theorem 6.1.** Suppose that a sequence of functions $\{f_n\}$ are $L^p$ integrable on a finite measure space, where $1 \leq p < \infty$. Then this sequence converges in $L^p$ to a measurable function $f$ if the following conditions are satisfied:

(i) $\{f_n\}$ converges to $f$ in measure; and

(ii) the functions $\{|f_n|^p\}$ are uniformly integrable.

**Remark 6.1.** One can easily prove for $p > 1$ and a nonempty family $X$ of random variables bounded in $L^p$ that if $\sup_{X \in X} ||X||_{L^p} < \infty$, then $X$ is uniformly integrable.

**Appendix B**

**Definition 6.1.** Suppose $(X, d)$ is a complete separable metric space with $\mathcal{B}(X)$ its associated Borel $\sigma$-algebra. Let $C_b(X)$ be the set of all real-valued continuous bounded functions on $X$, and let $Pr(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$. A collection $\Lambda \subset Pr(X)$ is tight if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that
\[
\mu(K_\epsilon) \geq 1 - \epsilon, \quad \forall \mu \in \Lambda. \tag{6.5}
\]

A sequence $\{\mu_n\}_{n \geq 0} \subset Pr(X)$ converges weakly to a probability measure $\mu$ if
\[
\int f \, d\mu_n \to \int f \, d\mu, \quad \forall f \in C_b(X). \tag{6.6}
\]

The proofs of the following results can be found in e.g. [DPZ92a].

**Proposition 6.1 (Prokhorov’s Theorem).** A collection $\Lambda \subset Pr(X)$ is weakly compact if and only if it is tight.

**Proposition 6.2 (Skorohod Representation Theorem).** Suppose that a sequence $\{\mu_n\}_{n \geq 0}$ converges weakly to a measure $\mu$. Then there exists a probability space $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}} \right)$ and a sequence of $X$-valued random variables $\{\tilde{Y}_n\}_{n \geq 0}$ relative to this space such that $\tilde{Y}_n$ converges a.s. to the random variable $\tilde{Y}$ and such that the laws of $\tilde{Y}_n$ and $\tilde{Y}$ are $\mu_n$ and $\mu$, respectively, i.e. $\mu_n(E) = \mathbb{P}(\tilde{Y}_n \in E), \mu(E) = \mathbb{P}(\tilde{Y} \in E), \forall E \in \mathcal{B}(X)$.

Finally, we suppose that $\{Y_n\}_{n \geq 0}$ is a sequence of $X$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mu_{m,n}\}_{m,n \geq 0}$ be the collection of joint laws of $\{Y_n\}_{n \geq 0}$, i.e.
\[
\mu_{m,n}(E) := \mathbb{P}((Y_m, Y_n) \in E), \quad \forall E \in \mathcal{B}(X \times X). \tag{6.7}
\]

We also need this result from [GK96]:
Proposition 6.3 (Gyöngy-Krylov Theorem). A sequence of $X$-valued random variables $\{Y_n\}_{n \geq 0}$ converges in probability if and only if for every subsequence of joint probability laws, $\{\mu_{mk,nk}\}_{k \geq 0}$ there exists a further subsequence which converges weakly to a probability measure $\mu$ such that

$$\mu(\{(x, y) \in X \times X : x = y\}) = 1. \quad (6.8)$$

The readers can find the detailed proof of the following lemma in [DGHT11].

Lemma 6.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space, $X$ a separable Hilbert space. Consider a sequence of stochastic bases $\mathcal{F}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_n^t\}_{t \geq 0}, \mathbb{P}, W^n)$, where $W^n$ is a cylindrical Brownian motion over $\Omega$ with respect to $\mathcal{F}_n^t$. Assume that $\{G^n\}_{n \geq 0}$ are a collection of $X$-valued $\mathcal{F}_n^t$ predictable processes such that $G^n \in L^2(0, T; L^2(\Omega, X))$ a.s. Finally, consider $\mathcal{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W_1, W_2)$ and $G \in L^2(0, T; L^2(\Omega, X))$ a.s., which is $\mathcal{F}_t$ predictable. If

$$G^n \to G \quad \text{in probability in } L^2(0, T; L^2(\Omega, X)), \quad (6.9)$$

$$W^n \to W \quad \text{in probability in } C([0, T]; \Omega_0), \quad (6.10)$$

then

$$\int_0^t G^n dW^n \to \int_0^t G dW \quad \text{in probability in } L^2(0, T; X). \quad (6.11)$$

Acknowledgments

This work was supported in part by NSF Grant DMS 1510249, and by the Research Fund of Indiana University.

References


[GK96] István Gyöngy and Nicolai Krylov. Existence of strong solutions for Itô’s stochastic equations


[TMF+12] Nakano T, Moore MJ, Wei F, Vasilakos AV, and Shuai J. Molecular communication and