Optimal Error Estimate of a Projection Based Interpolation for the $p$-Version Approximation in Three Dimensions

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Abstract—Optimal $p$-interpolation error estimate is derived for the local, projection-based interpolation for $H^1$-conforming elements in three space dimensions. Two different procedures leading to the same logarithmic term $\ln^{3/2} p$ in the estimate, are discussed © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

It is well known that the energy norm of the error of a conforming finite-element solution to an elliptic boundary value problem can be bounded by the distance from the exact solution to the approximation subspace of piecewise polynomials. This distance can be estimated by the difference between the exact solution and a suitable member from the subspace. Since the approximation subspaces are in general build from the polynomials defined over elements, the construction of the suitable approximation and its error estimate can be carried out over a single element. For various reasons, see e.g., [1, 2], the construction process needs to fulfill the following three conditions.

(i) Conformity The union of the element approximations must be globally conforming.

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(ii) **Optimality:** The approximation error should be of the same order as the best approximation error.

(iii) **Locality:** The restriction of the approximation to vertices, edges, and faces of an element should be determined only by the restriction of the solution to the corresponding part of the element.

For the h-version of the finite-element approximation, the approximation over each element is typically constructed by the Lagrange interpolation. This procedure produces a global approximation satisfying the above conditions for conformity and locality, and the error is also optimal with respect to the power of the element size. For the p-version approximation, due to the additional $p$ factors involved in the inverse inequality, the order (with respect to the polynomial degree $p$ used) of the piecewise Lagrange interpolation error is far from optimal. Optimal approximations in the $p$-version are usually constructed by the projection of the solution onto the approximation subspaces plus some modification on the element interfaces. There has been much work in this direction following the pioneering work of Babuška and Suri [3]. In [3], the approximation is constructed for two-dimensional problems by using the $H^1$-projection of the solution onto the polynomial space over each element and the modification is based on the $H^1$-projection of the residue on element edges. Muñoz-Sola [4] extended this idea to three-dimensional problems, which resulted in an approximation satisfying the three conditions mentioned above. However, since the $H^1$-projection is used for the modification on element edges and faces, the solution to be approximated must be in the Sobolev space $H^r(\Omega)$ with $r \geq 2$. This regularity requirement is not always satisfied, for instance, for problems with corner or edge singularities.

To relax the regularity restriction, Demkowicz [5] introduced the so-called **projection based interpolation.** The idea is to add the $H^1$-projection of the residual on the element edges and the $H^{1/2}(\partial K)$-projection of the residual on the element faces to the $H^1$-projection of the solution on the element, where $\epsilon$ is an arbitrary small positive number. The reason to use these projections is related to the trace theorem and the possibility of localizing boundary norms (the "localization theorem" in [5]). This procedure requires the solutions in $H^r(\Omega)$ with $r > 3/2$ only which, due to the Sobolev embedding theorem, is the weakest possible condition to have the locality condition satisfied. The convergence rate for the defined approximation is just a factor $p^2$ away from the optimal one.

The main difficulty encountered in constructing an $\epsilon$-free optimal approximation stems from two facts. First, the $H^{1/2}(\partial K)$-norm of the error over the boundary $\partial K$ of an element $K$ cannot be split into the $H^{1/2}(\partial f)$-norms over its faces $f$, see [6] for a counter example. The second difficulty is associated with the "break-down" of the trace theorem. The trace mapping from $H^{1/2}(f)$ to $L^2(\partial f)$ is no longer bounded, refer to [7, Theorem 9.5, pp. 43].

The purpose of this article is to construct an $\epsilon$-free optimal approximation for all $u \in H^r$ with $r > 3/2$. To overcome the difficulties mentioned above, two key observations are noted. First, it is possible to split the $H^{1/2}(\partial K)$-norm of a function in the polynomial subspaces, with a factor of order $\ln p$. Second, it is also possible to bound the $L^2(\partial f)$-norm by the $H^{1/2}(f)$-norm for polynomials, with a factor of order $(\ln p)^{1/2}$. Indeed, these tools have been established in a recent study by Cao and Guo [8] on the preconditioning techniques for the boundary element method in three dimensions.

With these tools at hand, we define the approximation on each element $K$, for any function $u \in H^r(K)$ with $r > 3/2$ by first taking the $H^1(K)$-projection of $u$, then adding to it the $L^2$-projection of the residual on each edge $e$, and the $H^{1/2}(\partial f)$-projection of the residual on each face $f$. This projection based interpolation procedure satisfies the conformity and locality. The convergence order is optimal with respect to the power of $p$, and the regularity requirement is the weakest.

An alternative procedure for eliminating the $\epsilon$-terms, was outlined by Demkowicz and Buffa [2]. The idea utilizes the fact that the blow up constants in both localization and trace theorems are
known in terms of $\epsilon$ and, ultimately, the term $O(\epsilon^{-r})p^{m\epsilon}$, with $r, m > 0$, can always be traded for term $\ln p$ in the final estimate. Both presented techniques yield identical results.

To simplify the presentation, we assume the partition of the solution domain is obtained with tetrahedral elements. Furthermore, since in the $p$-version of the finite-element method the partition is fixed, we may assume that each element $K$ is shape regular and of diameter $O(1)$. Throughout the paper, we use $\mathcal{P}_p(\omega)$ to denote the subspace of polynomials of total degree $\leq p$ over one-, two-, or three-dimensional domain $\omega$. $\mathcal{P}_p^0(\omega)$ represents the subspace of $\mathcal{P}_p(\omega)$ whose elements vanish at the boundary $\partial\omega$. We use $P^s$ to denote the orthogonal projection from $H^s(\omega)$ to $\mathcal{P}_p(\omega)$, and $P_0^s$ is the projection onto $\mathcal{P}_p^0(\omega)$. $C_0(\omega)$ will denote the space of continuous functions defined on one-, two-, or three-dimensional domain $\omega$, and vanishing on the boundary $\partial\omega$. We use $\Pi$ to represent the projection based interpolation. In addition, we shall use $c$ to represent a generic positive constant independent of both the polynomial degree $p$ and the function involved in the inequalities.

2. BASIC LEMMAS

We first define the projection based interpolation operator $\Pi_K : H^r(K) \to \mathcal{P}_p(K)$. Let $u \in H^r(K)$ with $r > 3/2$. $\Pi_K u$ is determined by the following three steps.

Step (i). For each edge $e$ of $K$, let $u_1$ be the linear interpolant of $u|_e$ at the end points of $e$. Let $P_0^0$ be the $L^2$-projection from $L^2(e)$ onto $\mathcal{P}_p^0(e) = \mathcal{P}_p(e) \cap C_0(e)$. Define

$$\Pi_e u = u_1 + P_0^0 (u - u_1).$$

Clearly, $\Pi_e u$ is the best polynomial approximation of $u|_e$ with respect to the $L^2$-norm, which coincides with $u$ at $\partial e$. Namely,

$$\|u - \Pi_e u\|_{0,e} \leq \|u - v\|_{0,e}, \quad \forall v \in \mathcal{P}_p(e) \text{ satisfying } v = u \text{ on } \partial e. \quad (1)$$

Step (ii). For each face $f$ of $K$, let $u_2 \in \mathcal{P}_p(f)$ be a polynomial extension from $L^2(\partial f)$ to $H^{1/2}(f)$ of the piecewise polynomials given by $\Pi_e u$ on all edges of $f$. The existence of such extensions is established in, e.g., [9,10]. Let $P_0^{1/2}$ be the $H^{1/2}$-projection from $H^{1/2}(f)$ onto $\mathcal{P}_p^0(f) = \mathcal{P}_p(f) \cap C_0(f)$. Define

$$\Pi_f u = u_2 + P_0^{1/2} (u - u_2).$$

Indeed, $\Pi_f u$ is the best polynomial approximation of $u|_f$ with respect to the $H^{1/2}$-norm, which coincides with $u_2$ at $\partial f$. That is,

$$\|u - \Pi_f u\|_{1/2,f} \leq \|u - v\|_{1/2,f}, \quad \forall v \in \mathcal{P}_p(f) \text{ satisfying } v = u_2 \text{ on } \partial f. \quad (2)$$

Step (iii). Finally, let $u_3 \in \mathcal{P}_p(K)$ be a polynomial extension from $H^{1/2}(\partial K)$ to $H^1(K)$ of the piecewise polynomials given by $\Pi_f u$ on all faces of $K$, see, e.g., [4]. Let $P_0^1$ be the $H^1$-projection from $H^1(K)$ onto $\mathcal{P}_p^0(K) = \mathcal{P}_p(K) \cap C_0(K)$. Define

$$\Pi_K u = u_3 + P_0^1 (u - u_3).$$

$\Pi_K u$ is also the best polynomial approximation of $u$ with respect to the $H^1$-norm which coincides with $u_3$ at $\partial K$.

$$\|u - \Pi_K u\|_{1,K} \leq \|u - v\|_{1,K}, \quad \forall v \in \mathcal{P}_p(K) \text{ satisfying } v = u_3 \text{ on } \partial K. \quad (3)$$
It is obvious that the restrictions of $\Pi_K u$ to the vertices, edges, and faces are determined uniquely by $u$ on the corresponding parts of the element. Thus, the interpolation procedure satisfies the conformity and locality conditions. This interpolation is also defined for any function in $H^s(K)$ with $r > 3/2$. This is the minimum requirement on $u$ to admit the pointwise values at the vertices of $K$.

REMARK 2.1. The described $H^1$-interpolation procedure belongs to a more general family of $H^1$, $H(\text{curl})$, and $H(\text{div})$ interpolation procedures studied in [2]. In order to enable commutativity properties of the interpolation operators, the projections in $H^{1/2}(\Omega)$ and $H^1(K)$ seminorms have to be replaced with projections in $H^{1/2}(\Omega)$ and $H^1(K)$ seminorms. Poincare's lemmas imply that both procedures admit identical error estimates, see [2] for details.

In order to derive the optimal error estimate for $\Pi_K u$, we begin with several lemmas. First, we state a conclusion about $I_N u$.

**LEMMA 2.1.** Let $I = [-1, 1]$ and $s > 1/2$. For any $v \in H^s(I)$, there exists a sequence $v_p \in \mathcal{P}_p(I)$ satisfying $v_p(\pm 1) = v(\pm 1)$, such that

$$||v - v_p||_{0,I} \leq c p^{-s} ||v||_{s,I},$$

where $c$ is independent of $v$ and $p$.

**PROOF.** Let $P^s$ be the orthogonal projection onto $\mathcal{P}_p$ with respect to the inner product for $H^s(I)$, see, e.g., [11]. Then, we have from Lemma 3.3 in [12] that for any $\mu \in [0, s]$ and for all $v \in H^s(I)$,

$$||v - P^s v||_{\mu,I} \leq c p^{-s} ||v||_{s,I}.$$  \hfill (4)

Setting $\mu = 0$ in the above inequality yields

$$||v - P^s v||_{0,I} \leq c p^{-s} ||v||_{s,I},$$

Since $s > 1/2$, we may also choose $\mu$ with $1/2 < \mu \leq \min(1, s)$ in (4). Then, it follows from the Sobolev embedding theorem that

$$||v - P^s v||_{L^\infty(I)} \leq c ||v - P^s v||_{\mu,I} \leq c p^{-s} ||v||_{s,I}.$$  \hfill (5)

Now, let $\phi_0$ and $\phi_1$ be the polynomials in $\mathcal{P}_p(I)$ of the least $L^2$-norm and satisfying

$$\phi_0(-1) = 1, \quad \phi_0(1) = 0,$$

$$\phi_1(-1) = 0, \quad \phi_1(1) = 1.$$  \hfill (6)

The explicit formula for $\phi_0$ and $\phi_1$ are given in Lemma 4.1 of [12]. It is shown there that

$$||\phi_0||_{0,I} = 2 \frac{1}{p(p+2)}, \quad i = 0, 1$$

We define

$$v_p = P^s v + (v - P^s v)(-1) \phi_0 + (v - P^s v)(1) \phi_1.$$  \hfill (7)

Then, clearly, we have $v_p \in \mathcal{P}_p$ satisfying $v_p(\pm 1) = v(\pm 1)$. Furthermore, by noting that $1/2 < \mu \leq 1$,

$$||v - v_p||_{0,I} \leq ||v - P^s v||_{0,I} + ||v - P^s v||_{L^\infty(I)} (||\phi_0||_{0,I} + ||\phi_p||_{0,I})$$

$$\leq c p^{-s} ||v||_{s,I} + c p^{-s} p^{-1} ||v||_{s,I}$$

$$\leq c p^{-s} ||v||_{s,I}.$$  \hfill (8)

From this lemma, we conclude immediately that for any $u \in H^s(\Omega)$ with $s > 1/2$,

$$||u - \Pi_\epsilon u||_{0,\epsilon} \leq c p^{-s} ||u||_{s,\epsilon}.$$  \hfill (9)

Next, we list some results about the error estimates for higher-order approximations in two and three dimensions.
LEMMA 2.2. Let $K$ be a tetrahedral element, and let $f$ be one of its faces and $e$ one of its edges. Then,

(i) for any $v \in H^r(K)$ with $r > 1$, there exists a sequence $v_p \in \mathcal{P}_p(K)$, such that

$$
\|v - v_p\|_{1,K} \leq cp^{-(r-1)} \|v\|_{r,K} ;
$$

(ii) for any $v \in H^r(f)$, there exists a sequence $v_p \in \mathcal{P}_p(f)$, such that

$$
\|v - v_p\|_{q,f} \leq cp^{-(r-q)} \|v\|_{r,f} , \quad \text{for } 0 \leq q \leq r,
$$

$$
\|v - v_p\|_{0,e} \leq cp^{-(r-(1/2))} \|v\|_{r,f} , \quad \text{for } r > \frac{1}{2},
$$

The first conclusion is a generalization of Lemma 11 in [4] to noninteger index $r$ obtained by interpolation of spaces. A conclusion similar to (ii) has been established in Lemma 3.1 of [3] for quadrilateral elements and the polynomials of separate degree $\leq p$. Note simply that $\mathcal{P}_p(f)$ contains the subspace of polynomials of total degree $\leq \lceil p/2 \rceil$. (ii) comes readily from applying Lemma 3.1 in [3] to the extension of $u$ to a quadrilateral.

Finally, we list two key lemmas used for overcoming the difficulties associated with the split-up of $H^{1/2}(\partial K)$-norm and the breakdown of the trace theorem from $H^{1/2}(f)$ to $L^2(\partial f)$.

LEMMA 2.3. Let $\partial K = \bigcup f_i$ be the boundary of an element $K$. There exists a positive constant $c$ depending only on the shape of $K$ such that for any $v \in \mathcal{P}_p(\partial K)$,

$$
\|v\|_{1/2,\partial K} \leq c \ln p \sum \|v\|_{1/2,f_i} .
$$

PROOF. It is shown in Lemma 4.8 in [8] that if $\omega$ is the union of a fixed number of triangles in $\mathbb{R}^2$, i.e., $\omega = \bigcup T_i$, then for any piecewise continuous polynomial $v$ on $\omega$,

$$
\|v\|_{1/2,\omega} \leq c \ln p \sum \|v\|_{1/2,T_i} ,
$$

where $c$ depends only on the shape of $T_i$. This proof is readily extended to the case where $\omega = \partial K$ is the union of triangles in $\mathbb{R}^3$.

This lemma can also be proved by using the following inequality,

$$
\|v\|_{1/2+e,\mathbb{R}^2} \leq ce^{-1} \left( \|v\|_{1/2+e,\mathbb{R}^2_+} + \|v\|_{1/2+e,\mathbb{R}^2_-} \right) , \quad \forall v \in H^{1/2}(\mathbb{R}^2) ,
$$

where $\mathbb{R}^2_+$, $\mathbb{R}^2_-$ are upper and lower half planes, see proofs on pages 29–30 of [13] and in Lemma 9 of [5]). Restricting $v$ to be in $\mathcal{P}_p(f)$ in the above inequality and using the inverse inequality,

$$
\|v\|_{1/2+e,f} \leq cp^{-2e} \|v\|_{1/2,f} ,
$$

we get the desired conclusion (9) by choosing $e = 1/\ln p$, compare the argument in [2].

LEMMA 2.4. Let $f$ be a face of $K$. There exists a positive constant $c$, independent of $p$, such that, for all $v \in \mathcal{P}_p(f)$

$$
\|v\|_{L^2(\partial f)} \leq c (\ln p)^{1/2} \|v\|_{1/2,f} .
$$

PROOF. See [8]. Again, an alternative reasoning can be offered by utilizing the inequality,

$$
\|v\|_{L^2(\partial f)} \leq c e^{-1/2} \|v\|_{1/2+e,f} ,
$$

(see e.g., [14, p. 100]), inverse estimate (10), and choice $e = 1/\ln p$. 

3. MAIN CONCLUSION

THEOREM 3.1. Let $K$ be a tetrahedral element and let $r > 3/2$. The projection based interpolation operator,

$$\Pi_K : H^r(K) \to \mathcal{P}_p(K),$$

admits the following estimates.

(i) $$\|u - \Pi_K u\|_{0,e} \leq c p^{-(r-1)} \|u\|_{r,K}, \quad (12)$$

(ii) $$\|u - \Pi_K u\|_{1/2,f} \leq c (\ln p)^{1/2} p^{-(r-1)} \|u\|_{r,K}, \quad (13)$$

(iii) $$\|u - \Pi_K u\|_{1,K} \leq c (\ln p)^{3/2} p^{-(r-1)} \|u\|_{r,K}. \quad (14)$$

PROOF The first conclusion comes directly from (5) and the definition of $\Pi_K u$ on $e$. Now, we show the second conclusion. Let $P^{1/2}u$ be the $H^{1/2}$-projection of $u$ onto $\mathcal{P}_p(f)$. Then,

$$\|u - \Pi_K u\|_{1/2,f} \leq \|u - P^{1/2}u\|_{1/2,f} + \|P^{1/2}u - \Pi_K u\|_{1/2,f}. \quad (15)$$

Note that $r > 3/2$. It follows from (8) that

$$\|u - P^{1/2}u\|_{1/2,f} \leq c p^{-(r-1)} \|u\|_{r-1/2,f}.$$

To deal with the second term in (15), we denote by $(\cdot, \cdot)_{1/2,f}$ the inner product of $H^{1/2}(f)$. It follows from the definition of $P^{1/2}$ on $f$,

$$(P^{1/2}u - u, v)_{1/2,f} = 0, \quad \forall v \in \mathcal{P}_p(f),$$

and from the definition of $\Pi_K$ on $f$,

$$(\Pi_K u - u, v)_{1/2,f} = (\Pi_K u - u_2 - (u - u_2), v)_{1/2,f} = (P^0_{1/2} (u - u_2) - (u - u_2), v)_{1/2,f} = 0, \quad \forall v \in \mathcal{P}^0_p(f).$$

The above two equations imply that

$$(P^{1/2}u - \Pi_K u, v)_{1/2,f} = 0, \quad \forall v \in \mathcal{P}^0_p(f).$$

Hence, we have, for any $v \in \mathcal{P}_p(f)$ that coincides with $P^{1/2}u - \Pi_K u$ on $\partial f$,

$$\|P^{1/2}u - \Pi_K u\|_{1/2,f} \leq \|v\|_{1/2,f}.$$

Denote by $E_0$ the extension operator from $L^2(\partial f)$ to $H^{1/2}(f)$ which maps piecewise continuous polynomials on $\partial f$ to a polynomial on $f$ of the same degree. Such an operator exists and is shown to be bounded in [9], see also [10] for an explicit construction of $E_0$. We have from the above inequality that

$$\|P^{1/2}u - \Pi_K u\|_{1/2,f} \leq \left\| E_0 \left( \left( P^{1/2}u \right)_{\partial f} - (\Pi_K u)_{\partial f} \right) \right\|_{1/2,f} \leq c \|P^{1/2}u - \Pi_K u\|_{0,\partial f}. $$
Let $z_p \in \mathcal{P}_p(f)$ be the function given in Lemma 2.2(i). It follows from the triangle inequality that

$$\|P^{1/2}u - \Pi_K u\|_{1/2, f} \leq c \left( \|P^{1/2}u - z_p\|_{0, \partial f} + \|\Pi_K u - z_p\|_{0, \partial f} \right). \quad (16)$$

The first term in the right-hand side above can be bounded by using (8) as

$$\|P^{1/2}u - z_p\|_{0, \partial f} \leq c (\ln p)^{1/2} \|P^{1/2}u - z_p\|_{1/2, f} \leq c (\ln p)^{1/2} \left[ \|u - P^{1/2}u\|_{1/2, f} + \|u - z_p\|_{1/2, f} \right] \leq c (\ln p)^{1/2} \left[ \|u\|_{r-1/2, f} \right].$$

For the second term in the right-hand side of (16), we have

$$\|z_p - \Pi_K u\|_{0, \partial f} \leq \|u - z_p\|_{0, \partial f} + \|u - \Pi_K u\|_{0, \partial f}, \quad (17)$$

where the first term can be bounded by $cp^{-(r-1)}\|u\|_{r-1/2, f}$, and the second term can be bounded by $cp^{-(r-1)}\|u\|_{r, \partial f}$ using Conclusion (i). Combining these estimates, we conclude (ii).

To establish Conclusion (iii), Let $P^1 : H^1(K) \rightarrow \mathcal{P}_p(K)$ be the orthogonal projection with respect to $H^1$-norm. It follows from the definition of $\Pi_K$ that

$$\|u - \Pi_K u\|_{1, K} \leq \|u - P^1 u\|_{1, K} + \|P^1 u - \Pi_K u\|_{1, K} \leq \|u - P^1 u\|_{1, K} + \|E_1\|_{H^{1/2}(\partial K), H^1(K)} \|P^1 u - \Pi_K u\|_{1/2, \partial K} \quad (18)$$

where $E_1$ is an extension operator from $H^{1/2}(\partial K)$ to $H^1(K)$ which maps $\mathcal{P}_p(\partial K)$ to $\mathcal{P}_p(K)$. Such an operator exists and is bounded, see Muñoz-Sola [4] By Lemma 2.3, the $H^{1/2}(\partial K)$-norm in the right hand side above can be split as follows,

$$\|P^1 u - \Pi_K u\|_{1/2, \partial K} \leq c \ln p \sum_f \|P^1 u - \Pi_K u\|_{1/2, f} \leq c \ln p \sum_f \left( \|u - P^1 u\|_{1/2, f} + \|u - \Pi_K u\|_{1/2, f} \right) \leq c \ln p \|u - P^1 u\|_{1/2, \partial K} + \ln p \sum_f \|u - \Pi_K u\|_{1/2, f} \leq c \ln p \|u - P^1 u\|_{K} + \ln p \sum_f \|u - \Pi_K u\|_{1/2, f}, \quad (19)$$

where we have used the trace theorem and the fact that

$$\sum_f \|v\|_{1/2, f} \leq c \|f\|_{1/2, \partial K}, \quad \forall v \in H^{1/2}(\partial K).$$

Putting (19) into (18), we have

$$\|u - \Pi_K u\|_{1, K} \leq c \ln p \|u - P^1 u\|_{K} + \ln p \sum_f \|u - \Pi_K u\|_{1/2, f},$$

The first term in the right-hand side above can be bounded by using (6), while the second terms can be bounded by using Conclusion (ii). This completes the proof of Conclusion (iii).
REMARK 3.1. An alternative proof of Estimate (iii) can again be offered by applying directly the argument from [2]. Two details need a clarification.

- The interpolation procedure has to be defined using $\epsilon$-free projections, i.e., the $L^2$-projections over edges, and the $H^{1/2}$-projections over faces.
- The contribution from the blow up constant in the trace theorem,

$$\|v\|_{t,e} \leq \epsilon^{-1/2} \|v\|_{1/2+t,f}, \quad \forall v \in H^{1/2+t}(f),$$

in Theorem 4 of [2] is missing. The result should be restated as

$$\|u - \Pi u\|_{1,K} \leq \epsilon^{-3/2} p^{-2(r-1/\epsilon)} \|u\|_{r,K}.$$ 

Here, $\epsilon^{-1}$ comes from the localization argument for the $H^{1/2}$ norm and $\epsilon^{-1/2}$ from the trace theorem above. Then, setting $\epsilon = 1/\ln p$ yields Estimate (iii).

Both types of argumentation yield the identical logarithmic term.

REMARK 3.2. The conclusion of this theorem is also valid in the case where $K$ is a nonsimplex element. Indeed, if the face $f$ of $K$ is a quadrilateral, then Lemma 2.2(i) holds too, and Lemma 2.4 has been established for quadrilaterals in Lemma 5 of [6]. For Lemma 2.3, it is always possible to express $\partial K$ as the union of triangles, thus, this lemma is still valid. The boundedness of the extension operators is established for quadrilaterals in [6].

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