On the superconvergence patch recovery techniques for the linear finite element approximation on anisotropic meshes

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A B S T R A C T

We provide in this paper an analysis on the superconvergence patch recovery (SPR) techniques for the linear finite element approximation based on adaptively refined anisotropic meshes in two dimensions. These techniques include the gradient recovery based on local weighted averaging, the recovery based on local $L^2$-projection, and the recovery based on least square fitting. The last one leads to the Zienkiewicz–Zhu type error estimators popular in engineering communities. Based on the superconvergence result for anisotropic meshes established recently in Cao (2013), we prove that all three types of SPR techniques produce super-linearly convergent gradients if the meshes are quasi-uniform under a given metric and each pair of adjacent elements in the meshes form an approximate parallelogram. As a consequence, the error estimators based on the recovered gradient are asymptotically exact. These results provide a theoretical justification for the extraordinary robustness and accuracy observed in numerous applications for the recovery type error estimators on anisotropic meshes.

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1. Introduction

For problems with local anisotropic features, such as boundary and internal layers, finite element (FE) approximation based on adaptive anisotropic meshes can be much more efficient than the one based on shape regular meshes. Over the last decade, there has been a growing interest in the analysis and application of the finite element method (FEM) based on anisotropic meshes, see, e.g., [1–13]. An important component of the adaptive FE solution process is the a-posteriori error estimation, which is used to assess the accuracy of the numerical solutions and to guide the adaptive mesh refinements. There are mainly two types of a-posteriori error estimate techniques, residual based estimates and recovery based estimates. Both types of techniques are well established for the FE approximation on shape regular meshes, see, e.g., the monographs by Ainsworth and Oden [14], Babuška and Strouboulis [15], Wahlbin [16], and Zhu [17]. For the FE approximation on anisotropic meshes, there have been much research on the residual based error estimate techniques in recent years. For instance, Kunert and Nicaise [10,11] proved both upper and lower bounds for various residual based error estimators for two and three dimensional problems. Formaggia [18] studied the residual based error estimators for advection–diffusion–reaction problems and Stokes problems. Creusé and Nicaise [4], and Houston [9] provided an analysis on the residual based error estimators for the discontinuous Galerkin method on anisotropic meshes.
On the other hand, there have also been many reports on the numerical study of the recovery type error estimates on anisotropic meshes, including the prominent Zienkiewicz–Zhu (ZZ) type [19,20] error estimators in particular [3,5,13, 21–25]. Numerous examples and applications demonstrate that recovery type error estimators behave extraordinarily well even at the presence of highly anisotropic elements. They are not only reliable and efficient, but also often asymptotically exact. These features make recovery type estimators a favorable choice for FE practitioners in engineering communities. However, there has been very few theoretical work justifying these good behaviors in the case of anisotropic adaptive meshes. It is well-known that for the FEM based on shape regular meshes, superconvergence of the finite element solution plays an indispensable role in the analysis on the recovery type error estimators. This theory is still under development for anisotropic meshes. There have been a few superconvergence studies on the FE approximation on anisotropic meshes. But they mostly deal with uniform meshes or structured meshes [26–29]. Superconvergence for linear FE approximation on general adaptively refined anisotropic meshes has not been established until very recently in [30].

In [30] we considered the finite element approximation based on a class of meshes that are quasi-uniform under a given metric. By extending the concept of approximate parallelogram introduced in [31,32] for shape regular meshes to anisotropic meshes, we established the super-linear convergence of the finite element solution to the interpolation of the exact solution in $H^1$-norm. We also obtained the super-linear convergence for a gradient recovery based on the global $L^2$-projection of gradient of the finite element solution. It implies that the error estimator based on the global $L^2$-projection recovery is asymptotically exact for the adaptive anisotropic meshes.

The purpose of this paper is to provide an analysis on the superconvergence patch recovery (SPR) techniques, which are based on local instead of global operations, for the linear FEM on anisotropic meshes in two dimensions. More precisely, we study three types of gradient recovery in the SPR family. They include the recovery based on local weighted averaging, the recovery based on local $L^2$-projection, and the recovery based on least square fitting. The third one leads to the well-known ZZ-type error estimators. Based on the superconvergence result established in [30] and following similar ideas as in [33,34], we prove that all three types of SPR recovery techniques produce super-linearly convergent gradient if the meshes are quasi-uniform under a given metric and each pair of adjacent elements in the meshes form an approximate parallelogram.

As a consequence, the error estimators based on the SPR gradient recovery are asymptotically exact. These results provide a theoretical justification for the extraordinary robustness and accuracy observed in numerous applications for the recovery type error estimators on anisotropic meshes.

A sketch of the paper is as follows: In Section 2 we first describe the FE approximation based on a class of anisotropic meshes that are quasi-uniform under a metric. Then we summarize the superconvergence of the FE solution established in [30]. In particular, we recall the concept of approximate parallelograms introduced there, which is also essential for our analysis of the SPR techniques. In Section 3 we develop the analysis on the recovery techniques based on local weighted averaging, local $L^2$-projection, and least square fitting, separately. We present in Section 4 a numerical example to demonstrate the superconvergence of the recovery techniques and the asymptotic exactness of the error estimators. And we conclude the paper with some comments in Section 5.

Throughout this paper, we use $c$ to represent a general positive constant independent of the mesh and the functions involved, and use \( \sim \) to represent the quantities on each side of it are equivalent with equivalence constants independent of the mesh and functions involved. For vectors we typically use $\| \cdot \|$ for their Euclidean norms, and for matrices we use $\| \cdot \|$ for their $2$ -norms. For functions defined on domain $D$, we use $\| \cdot \|_D$ and $\| \cdot \|_{1,D}$ to represent, respectively, their $L^2$-norm and $H^1$-norm over $D$. When $D$ is the entire domain for the PDE, we may omit the subscript $D$ in the norms.

2. Model PDE and its FE approximation based on anisotropic meshes

We consider the following homogeneous Dirichlet problem of a second order elliptic equation:

\[
\begin{aligned}
\begin{cases}
-\nabla \cdot (A \nabla u + bu) + du & = f, & \text{in } \Omega \\
|u|_{\partial \Omega} & = 0,
\end{cases}
\end{aligned}
\]

where $A$ is a positive definite constant matrix, $b$, $d$, and $f$ are suitably smooth functions. Eq. (1) is assumed to be strongly elliptic.

**FE approximation:** Note that for discretization involving anisotropic meshes, the element diameter is no longer a proper parameter for describing the asymptotic behaviors, since each element can have much different length scales in different directions. Instead, the total number $N$ of elements should be used to characterize the fineness of the discretization. Thus we use $\{T_N\}$ to represent a family of triangulations of $\Omega$ satisfying the basic requirement that the intersection of the closures of any two elements is either the empty set, a point, or an entire edge. Define $S_N$ be the space of continuous piecewise linear functions based on partition $T_N$. $V_N = S_N \cap H_0^1(\Omega)$. The finite element method for solving (1) is to find the approximate solution $u_N \in V_N$ satisfying

\[
\int_{\Omega} [(A \nabla u_N) \cdot \nabla v + u_N (b \cdot \nabla v) + d u_N v] = 0, \quad \forall v \in V_N.
\]

In order to better describe and control the anisotropic mesh features, such as element sizes, aspect ratios, and alignment directions, we restrict our analysis to a class of meshes that are quasi-uniform under a given metric. Let $M$ be a continuous
Riemannian metric on \( \tilde{\Omega} \). For each element \( \tau \in \mathcal{T}_N \), let \( M_\tau \) be the average of \( M \) over \( \tau \). Its eigen-decomposition is of form

\[
M_\tau = T_\tau \cdot \Lambda_\tau \cdot T_\tau^T,
\]

where \( \Lambda_\tau \) is diagonal and \( T_\tau \) is orthonormal. Define

\[
F_\tau = T_\tau \Lambda_\tau^{-\frac{1}{2}}.
\]

We call a family of triangulations \( \{ \mathcal{T}_n \} \) quasi-uniform under metric \( M \), if for all \( \tau \in \mathcal{T}_N \), \( \tilde{\tau} = F_\tau^{-1} \tau \) are shape regular and of about the same size, see [35–37]. Let \( J_\tau \) be the Jacobian of the affine mapping from a standard element \( \tilde{\tau} \) to \( \tau \). Then the fact that \( \{ \mathcal{T}_n \} \) is quasi-uniform under metric \( M \) can be characterized by

\[
\|F_\tau^{-1}J_\tau\| \simeq (\|J_\tau^{-1}F_\tau\|)^{-1} \simeq (CM/N)^{1/2}, \quad \forall \tau \in \mathcal{T}_N,
\]

where

\[
CM = \int_{\Omega} |\det(M)|^{1/2}.
\]

In order to derive the convergence and superconvergence analysis for the linear FE approximation, we need the following assumption to characterize the anisotropic behavior of the second and third order derivatives of solution \( u \).

**Assumption on \( D^2u \) and \( D^3u \):** Let \( Q_2(\mathbf{x}) \) and \( Q_3(\mathbf{x}) \) be \( 2 \times 2 \) symmetric non-negative definite matrices. For \( m = 2, 3 \), we assume at each \( \mathbf{x} \in \Omega \) that

\[
|(\mathbf{s} \cdot \nabla)^m u(\mathbf{x})| \leq |\mathbf{s} \cdot Q_m(\mathbf{x})\mathbf{s}|^{m/2}, \quad \forall \mathbf{s} \in \mathbb{R}^2.
\]

Such matrices always exist as long as \( D^2u \) and \( D^3u \) exist. For instance, \( Q_2(\mathbf{x}) \) can be chosen as \( \|\nabla^2 u(\mathbf{x})\|I \), where \( I \) is the identity matrix, and \( \|\nabla^2 u(\mathbf{x})\| \) is the maximum of all the second order directional derivatives at \( \mathbf{x} \). It can also be chosen as \( \operatorname{abs}(\nabla^2 u(\mathbf{x})) \), which is the matrix of the same eigenvectors as \( \nabla^2 u(\mathbf{x}) \) but with the eigenvalues equal to the absolute values of \( \nabla^2 u(\mathbf{x}) \)'s. Note that the left hand side of (7) is the directional derivative along \( \mathbf{s} \), the “smaller” the matrix \( Q_m \) is, the more precisely it describes the anisotropic behaviors of \( D^m u \), and the tighter the error bounds for both a priori and a-posteriori error estimates based on it. More details on how to find these “anisotropic measures” \( Q_m \) numerically for \( m \geq 3 \) can be found in [35,36]. More recently, Mirebeau discussed the basic properties of \( Q_m \) for general \( m \) [38], and derived in particular an explicit formula for the “smallest” \( Q_2(\mathbf{x}) \) measuring the anisotropic behaviors of third order derivatives in his dissertation [39].

The advantages of using anisotropic meshes in the FE solution of PDEs with anisotropic characters can be appreciated by the following a-priori error estimate established in [30] for the linear FE solution of (1).

**Theorem 2.1.** Suppose \( \{ \mathcal{T}_N \} \) is quasi-uniform under metric \( M \), and \( u_N \) is the linear finite element approximation to the solution \( u \) of (1). Furthermore, assume the second and third derivative \( D^2u \) satisfies assumption (7) about its anisotropic behavior. Then

\[
\|u - u_N\|_{1, \Omega} \leq cN^{-1/2} \left\{ C_M \int_{\Omega} \|F^{-1}\|^2 \|F^T Q_2 F\|^2 \right\}^{1/2},
\]

where \( F \) and \( C_M \) are determined by \( M \) in (4) and (6).

When \( \nabla^2 u \) is positive definite on \( \Omega \), we may choose \( Q_2 = \nabla^2 u \) and \( M = c(\lambda_{\text{max}}/\lambda_{\text{min}})^{1/4} \cdot \nabla^2 u \), where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the largest and smallest eigenvalues of \( \nabla^2 u \), respectively. In this case, (8) is reduced to

\[
\|u - u_N\|_{1, \Omega} \leq cN^{-1/2} \|\lambda_{\text{min}}^{1/4}/\lambda_{\text{max}}^{1/4}\|_{L^1(\Omega)}.
\]

If we allow only shape regular adaptive meshes, then the best isotropic mesh metric to minimize the error bound in (8) would be \( M = \lambda_{\text{max}} I \), and the error bound for \( |u - u_N|_{1, \Omega} \) would be \( cN^{-1/2} \|\lambda_{\text{max}}\|_{L^1(\Omega)} \). This is clearly much larger than that for the anisotropic case when \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are of very different magnitudes.

**Superconvergence of FE solution:** Local mesh symmetry plays an essential role in the superconvergence analysis of the FE approximation. For linear elements, this symmetry typically requires each pair of elements sharing a common edge to form a parallelogram. For applications with non-uniform meshes, a concept of \( O(h^{1+\alpha}) \)-approximate parallelogram was introduced in [17,31,32,34,40] for the analysis of superconvergence properties on “mildly structured” quasi-uniform meshes. This notion was further extended in [30] to non-uniform, unstructured anisotropic meshes. We recall here the so-called \( O(N^{-(1+\alpha)/2}) \)-approximate parallelograms introduced in [30].

**Definition 2.1.** Let the standard element \( \tilde{\tau} \) be an equilateral triangle with vertices on the unit circle. Let \( \tau_1 \) and \( \tau_2 \) be a pair of elements sharing a common edge, and let \( J_1 \) and \( J_2 \) be, respectively, the Jacobians of the affine mappings from \( \tilde{\tau} \) to \( \tau_1 \) and \( \tau_2 \) that map a vertex of \( \tilde{\tau} \) to the opposite vertices of \( \tau_1 \cup \tau_2 \). We call \( \tau_1 \cup \tau_2 \) forms an \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram, if

\[
\|I + J_1^{-1}J_2\| = O(N^{-\alpha/2}).
\]
In the special case \( \{ T_h \} \) is quasi-uniform, we have for the characteristic element diameter \( h = O(N^{-1/2}) \), and the above definition is equivalent to the definition of \( O(\varepsilon^{1+\alpha}) \)-approximate parallelograms for “mildly structured” and unstructured quasi-uniform meshes described in \([31,32,34]\). Indeed, anisotropic elements \( \tau_1 \) and \( \tau_2 \) form an \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram, iff they form a shape regular \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram under an affine mapping, see Lemma 2.2 in \([30]\). Also, if the two elements form an approximate parallelogram, then the ratio of their areas deviates from 1 by at most \( O(N^{-\alpha/2}) \), i.e.,

\[
\frac{|\tau_2|}{|\tau_1|} = 1 + O(N^{-\alpha/2}).
\]

Approximate parallelograms can also be characterized by the vector connecting the midpoints of its two diagonals. Let \( Q = \tau_1 \cup \tau_2 \), and let \( \hat{Q} \) be the standard quadrilateral element, e.g., \([0,1]^2 \). \( J_Q \) is the Jacobian of the bilinear mapping from \( \hat{Q} \) to \( Q \), and \( d \) is the vector connecting the midpoints of the diagonals of \( Q \). Then \( \tau_1 \cup \tau_2 \) forms an \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram, iff (cf. Lemma 2.3 in \([30]\))

\[
J_Q^{-1}(\xi)d = O(N^{-\alpha/2}), \quad \forall \xi \in \hat{Q}.
\]

Notice that when \( \tau_1 \cup \tau_2 \) forms an approximate parallelogram, both Jacobian \( J_1 \) and \( J_2 \) for the affine mapping from \( \hat{r} \) to \( \tau_1 \) and \( \tau_2 \) are equivalent to \( J_Q \), namely, \( \text{cond}(J_1J_2) = O(1) \). Thus \( (11) \) implies furthermore that

\[
|J_i^{-1}d| = O(N^{-\alpha/2}), \quad \forall i = 1, 2.
\]

For the linear FE approximation of \( (1) \) based on anisotropic meshes, we established in \([30]\) the following superconvergence for the FE solution to the interpolation of the exact solution in \( H^1 \)-seminorm.

**Theorem 2.2.** Let \( u_I \) and \( u_N \) be the linear interpolation and the finite element approximation of \( u \), respectively. Suppose \( D^2u \) and \( D^3u \) satisfy assumption \( (7) \), and each pair of adjacent elements in \( T_h \) form an \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram. Then

\[
\|u_N - u_I\|_{1,\Omega} \leq c \left\{ \sum_{\tau} \left( \int_\Omega (1 + N^{-\alpha} ||J_1^{-1}\|_{F}^2) \cdot \|J_1^TQJ_2\|_{F}^2 + \|J_2^{-1}\|_{F}^4 \cdot \|J_1^TQJ_2\|_{F}^2 \cdot \|J_2^{-1}\|_{F}^2 \cdot \|J_1^TQJ_2\|_{F}^3 \right) \right\}^{1/2}.
\]

In addition, if the partition \( T_h \) is quasi-uniform under metric \( M \), then we have

\[
\|u_N - u_I\|_{1,\Omega} \leq cN^{-1/2} \left\{ \int_\Omega ((C_M)^2N^{-1} + C_MN^{-\alpha}\|F^{-1}\|_{F}^4\|F\|_{F}^2) \cdot \|F^TQ_2F\|_{F}^2 \right. \\
\left. + (C_M)^2N^{-1}\|F^{-1}\|_{F}^4\|F\|_{F}^2 \cdot \|F^TQ_2F\|_{F}^3 \right\}^{1/2},
\]

where \( F \) and \( C_M \) are determined by \( M \) as in \( (4) \) and \( (6) \).

Based on this theorem, we established in \([30]\) the superconvergence of the gradient recovery using the global \( L^2 \)-projection of the finite element solutions. Here we shall extend this study to the recovery techniques based on local patch operations.

### 3. Analysis of superconvergence patch recovery techniques

In this section, we present our analysis on three superconvergence patch recovery (SPR) techniques, namely, the recovery based on local weighted averaging, local \( L^2 \)-projection, and local least square fitting. First we list several lemmas needed for the analysis.

**Lemma 3.1.** Suppose function \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( k \)-th order differentiable at an \( x \) in its domain, and suppose

\[
|\xi \cdot \nabla|^k u(x) \leq |\xi \cdot Q\xi|^{k/2}, \quad \forall \xi \in \mathbb{R}^n,
\]

for an \( n \times n \) symmetric positive definite (SPD) matrix \( Q \). Then we have for any \( \xi_1, \xi_2, \ldots, \xi_k \in \mathbb{R}^n \) that

\[
|\xi_1 \cdot \nabla)(\xi_2 \cdot \nabla) \cdots (\xi_k \cdot \nabla)u(x)| \leq c(n, k) \prod_{j=1}^k |\xi_j \cdot Q\xi_j|^{1/2},
\]

where \( c(n, k) \) is a constant depending only on \( n \) and \( k \). In particular, we have \( c(n, k) = 1 \) for \( k = 2 \).
which is equivalent to that for all \(\xi \in \mathbb{R}^n\) with \(|\xi| = 1\),
\[
|\xi \cdot \nabla u| \leq 1.
\]
(17)

Also, conclusion (16) is equivalent to the fact that for all \(\xi_i \in \mathbb{R}^n\) with \(|\xi_i| = 1\),
\[
|\xi_1 \cdot \nabla \xi_2 \cdot \nabla \cdots \cdot \xi_k \cdot \nabla u| \leq c(n, k).
\]
(18)

Note (17) implies that all the \(k\)-th order directional derivatives of \(u\) at \(\bar{x}\) are bounded from the above by 1. According to Lemma 2.2 in [36], we have all the mixed derivatives \(\partial_{i_1, i_2, \ldots, i_k} u\) are bounded by certain constant \(c\) depending only on \(n\) and \(k\). Finally, (18) follows from the fact that \((\xi_1 \cdot \nabla)(\xi_2 \cdot \nabla) \cdots (\xi_k \cdot \nabla)\) is a linear combination of these mixed derivatives with combination coefficients bounded by constants depending only on \(n\) and \(k\).

For the special case \(k = 2\), we may take advantage of the fact that \(\nabla^2 u\) is a symmetric matrix, which admits a real valued eigen-decomposition. Then (17) implies the absolute values of the eigenvalues of \(\nabla^2 u\) are bounded from the above by 1, and (18) follows easily from the Cauchy–Schwarz inequality with \(c(n, k) = 1\). \(\square\)

**Remark 3.1.** This lemma asserts that an elliptic anisotropic bound for the higher order directional derivatives can also provide a bound for the higher order mixed derivatives. We conjecture the constant \(c(n, k)\) in (16) is also 1 for any dimension \(n\) and any derivative order \(k\). Note that \(x\) here is fixed, this conjecture can also be phrased in pure algebraic terms. In particular, in the special case \(n = 2\), it implies that if a homogeneous polynomial of degree \(k\) takes 1 or \(-1\) as its extreme value over the unit circle, then the absolute value of its coefficient for term \(x^n y^{k-i}\) must be \(\leq \frac{1}{k}\). However, we do not know a proof of it yet.

**Lemma 3.2.** Suppose \(z = 0\) is an interior node, and \(\omega_z = \bigcup_{i=1}^m \tau_i\) is the element patch around \(z\). The vertices of \(\tau_i\) are \(z, x_{i-1}\) and \(x_i\) (with the index moduled by \(m\), see Fig. 1. Then

\[
\nabla u(z) = -\frac{1}{2|\omega_z|} R_z^T \sum_{i=1}^m (x_i \cdot \nabla u(z)) (x_{i+1} - x_{i-1}),
\]
(19)

where \(R_z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\) is the matrix of rotation by angle \(\tau\) counter-clock-wise.

**Proof.** By elementary calculation

\[
x_i^T (x_{i+1} - x_{i-1}) = \det([x_{i-1}, x_i]) R_z^T = 2|\tau_i| \cdot R_z^T.
\]

Thus we have

\[
\sum_i (x_i \cdot s) (x_{i+1} - x_{i-1}) = \sum_i (x_{i+1} x_i^T - x_{i-1} x_i^T) s
\]
\[
= \sum_i (x_i x_i^T - x_{i-1} x_i^T) s
\]
\[
= 2 \sum_i |\tau_i| R_z^T s
\]
\[
= 2|\omega_z| R_z^T s. \quad \square
\]
Lemma 3.3. Let $\omega_2 = \bigcup_{i=1}^m \tau_i$ be a patch of elements around an interior node $z$ as in Fig. 1, and assume each pair of adjacent elements $\tau_i$ and $\tau_{i+1}$ in $\omega_2$ form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. Let $F_i$ be an affine mapping from the equilateral standard element $\tilde{\tau}$ to $\tau_i$ and $J_i$ be its Jacobian. Then

(i) $\mbox{cond}(J_i^{-1}J_i) = O(1)$, for any $1 \leq i, i' \leq m$;

(ii) For any $i = 1, \ldots, m$, we have

$$|J_i^{-1}\sum_{i=1}^m x_i| = O(N^{-\alpha/2});$$

(iii) For any $i, i' \neq 2$, $F_i^{-1}(\omega_2)$ is a patch of shape regular triangles of diameter $O(1)$. Each adjacent element pair in $\hat{\omega}_2$ forms an $O(N^{-(1+\alpha)/2})$-approximate parallelogram.

Proof. We start with (i) for $i = 1$ and $i' = 2$. Note that there exist rotations $R_1$ and $R_2$ (by angle $0^\circ$ or $\pm 60^\circ$) such that $F_1 \circ R_1$ and $F_2 \circ R_2$ map $\tilde{\tau}$ onto $\tau_1$ and $\tau_2$, respectively, and in particular map $(1, 0) \in \tilde{\tau}$ to the opposite vertices in $\tau_1 \cup \tau_2$. Since $\tau_1 \cup \tau_2$ forms an $O(N^{-(1+\alpha)/2})$-approximate parallelogram, it follows from (9) that

$$\|I + (J_1 \circ R_1)^{-1}(J_2 \circ R_2)\| = O(N^{-\alpha/2}),$$

which implies

$$\mbox{cond}((J_1 \circ R_1)^{-1}(J_2 \circ R_2)) = O(1).$$

Since multiplication by a rotation matrix (from either left or right) does not change the 2-norm of a matrix, we have

$$\mbox{cond}(J_1^{-1}J_2) = O(1).$$

Similarly,

$$\mbox{cond}(J_2^{-1}J_3) = O(1).$$

Thus

$$\mbox{cond}(J_1^{-1}J_2) \leq \mbox{cond}(J_1^{-1}J_2) \cdot \mbox{cond}(J_2^{-1}J_3) = O(1).$$

Since $m$ is bounded, (20) follows by repeating the above arguments.

Next, since each $\tau_i \cup \tau_{i+1}$ forms an $O(N^{-(1+\alpha)/2})$-approximate parallelogram, we have from (12) that

$$|J_i^{-1}(x_{i+1} + x_{i-1} - x_i - z)| = O(N^{-\alpha/2}).$$

It follows from (20) that

$$|J_i^{-1}(x_{i+1} + x_{i-1} - x_i - z)| = O(N^{-\alpha/2}).$$

Summing up the above equation for all $i = 1, 2, \ldots, m$, we have

$$|J_i^{-1}\sum_{i=1}^m (x_i - z)| = |J_i^{-1}\sum_{i=1}^m (x_{i+1} + x_{i-1} - x_i - z)| = O(N^{-\alpha/2}).$$

Finally, note that (20) implies that each $F_i^{-1}(\tau_i)$ is shape regular and of diameter $O(1)$. The adjacent pairs $F_i^{-1}(\tau_i \cup \tau_{i+1})$ form approximate parallelograms by Lemma 2.2 in [30].

3.1. Recovery based on local weighted averaging

We consider first the post-processing procedure based on a weighted average of the gradient over element patches. Let $u_N$ be a piecewise constant function, and define $A[\nabla u_N]$ be the continuous piecewise linear function which assumes the following value at each node $z$ in the partition,

$$A[u_N](z) = \sum_{i=1}^m |\tau_i| \omega_2 |u_N|_{\tau_i},$$

where $\omega_2 = \bigcup_{i=1}^m \tau_i$ is the patch of elements around $z$, and $|\omega_2| = \sum_{i=1}^m |\tau_i|$ is its measure. The recovery of $\nabla u$ based on local weighted averaging then can be described as follows:

$$A[\nabla u_N] = \begin{bmatrix} A[\partial_1u_N] \\ A[\partial_2u_N] \end{bmatrix}.$$}

We shall prove a superconvergence of $A[\nabla u]$ to $\nabla u$ under similar assumptions as in Theorem 2.2. To this end, we first list a lemma based mainly on the geometry of $\omega_2$. 


Lemma 3.4. Suppose \( z = 0 \) is an interior node, and each pair of adjacent elements in patch \( \omega_2 \) form an \( O(N^{-1+\alpha}/2) \)-approximate parallelogram. Moreover, assume \( \nabla^2 u(z) \) satisfies (7) about its anisotropic behavior. Then
\[
\left| \sum_{j=1}^{m} |x_j^T \nabla^2 u(z) x_j| (x_{i+1} - x_{i-1}) \right| \leq c N^{-\alpha} \left( \sum_{j=1}^{m} \|f_j^T Q_2(x_{j})\| \right) \cdot \left( \sum_{j=1}^{m} \|f_j\| \right),
\]
(22)
where \( f_j \) is the Jacobian of the affine mapping from the standard element to \( \tau_i \).

**Proof.** For simplifying notation, we denote by \( H \) the Hessian matrix \( \nabla^2 u(z) \). Clearly
\[
\sum_i \left( x_i^T H x_i \right) (x_{i+1} - x_{i-1}) = \frac{1}{2} \sum_i \left[ \left( x_i^T H x_i \right) (x_{i+1} - x_{i-1}) + (x_{i+3}^T H x_{i+3}) (x_{i+4} - x_{i+2}) \right].
\]

Let
\[
d_i = \frac{1}{2} (x_{i+1} + x_{i-1}) - \frac{1}{2} x_i
\]
be the vector connecting the midpoints of two diagonals of \( \tau_i \cup \tau_{i+1} \). Then we have
\[
x_{i+2} + x_{i-1} = 2(d_{i+1} + d_i),
x_{i+4} + x_{i+1} = 2(d_{i+3} + d_{i+2}),
\]
and
\[
x_{i+4} - x_{i+2} = -(x_{i+1} - x_{i-1}) + 2(d_{i+3} + d_{i+2} - d_{i+1} - d_i).
\]
Hence,
\[
\left| (x_i^T H x_i) (x_{i+1} - x_{i-1}) + (x_{i+3}^T H x_{i+3}) (x_{i+4} - x_{i+2}) \right| \leq \left| (x_i^T H x_i - x_{i+3}^T H x_{i+3}) (x_{i+1} - x_{i-1}) \right|
\]
\[
+ 2 \left| (x_{i+3}^T H x_{i+3}) (d_{i+1} + d_{i+2} - d_{i+1} - d_i) \right|
\]
\[
\leq |x_i^T H x_i - x_{i+3}^T H x_{i+3}| \cdot |x_{i+1} - x_{i-1}|
\]
\[
+ 2 |x_{i+3}^T H x_{i+3}| \cdot |d_{i+1} + d_{i+2} - d_{i+1} - d_i|.
\]
Moreover, we have from \( x_{i+3} + x_i = 2(d_{i+1} + d_{i+2}) \) that
\[
\left| (x_i^T H x_i) (x_{i+1} - x_{i-1}) + (x_{i+3}^T H x_{i+3}) (x_{i+4} - x_{i+2}) \right| \leq 4 \left| (x_i - d_i - d_{i+2})^T H (d_i + d_{i+2}) \cdot |x_{i+1} - x_{i-1}| \right|
\]
\[
+ 2 |x_{i+3}^T H x_{i+3}| \cdot |d_{i+1} + d_{i+2} - d_{i+1} - d_i|.
\]
Since \( H = \nabla^2 u(z) \) satisfies (7), it follows from Lemma 3.1 that
\[
|x_i^T H d_i| \leq c \|x_i^T Q_2 x_i\|^{1/2} \cdot \|d_i^T Q_2 d_i\|^{1/2}
\]
\[
\leq \|f_j^T Q_2 f_j\|^{1/2} \cdot \left| x_i \right| \cdot \|f_j^T Q_2 f_j\|^{1/2} \cdot |d_i|.
\]
By using the assumption that \( \tau_i \cup \tau_{i+1} \) forms an \( O(N^{-1+\alpha}/2) \)-approximate parallelogram, we have from (12) that \( |J^{-1} d_i| = O(N^{-\alpha/2}) \), which implies
\[
|x_i^T H d_i| \leq c N^{-\alpha/2} \|f_j^T Q_2 f_j\|.
\]
Therefore,
\[
\left| (x_i - d_i - d_{i+2})^T H (d_i + d_{i+2}) \cdot |x_{i+1} - x_{i-1}| \right| \leq c N^{-\alpha/2} \cdot \left( \|f_j^T Q_2 f_j\| + \|f_j^T Q_2 f_{j+2}\| \right) \cdot \left( \|J^{-1}_j\| + \|J^{-1}_{j+1}\| \right),
\]
and
\[
|x_{i+3}^T H x_{i+3}| \cdot |d_{i+1} + d_{i+2} - d_{i+1} - d_i| \leq c N^{-\alpha/2} \cdot \|f_j^T Q_2 f_{j+3}\| \cdot \left( \|J^{-1}_{j+3}\| + \|J^{-1}_{j+2}\| + \|J^{-1}_{j+1}\| + \|J^{-1}_j\| \right),
\]
which complete the proof of (22). \( \Box \)

Lemma 3.5. Suppose \( z \) is an interior node, and each pair of adjacent elements in \( \omega_2 \) form an \( O(N^{-1+\alpha}/2) \)-approximate parallelogram. Furthermore, assume \( u \) satisfies (7) about its anisotropic behavior. Then we have
\[
|d_{\mathcal{A}}(\nabla u_1)(z) - \nabla u(z)| \leq c \max_{\tau \in \omega_2} \|J^{-1}_\tau\| \left( \|J_{\tau}^T Q_2 J_{\tau}\| \right)^{3/2},
\]
(23)
**Proof.** Since $\mathcal{A}[\nabla u_1](z)$ is invariant with respect to the translation of $\omega_2$, we may assume $z = 0$. Also we assume the nodes $\mathbf{z}, \mathbf{x}_1, \ldots, \mathbf{x}_m$ are arranged as in Fig. 1. Let $u_i = u(\mathbf{x}_i)$, and let $\phi_i$ be the nodal basis function associated with $\mathbf{x}_i$. Then

$$\nabla u_i|_{\tau} = u(\mathbf{z}) \nabla \phi_z + u_i \nabla \phi_i + u_{i-1} \nabla \phi_{i-1}$$

$$= \frac{-1}{2|\omega_2|} \cdot R^2 \sum_{i=1}^{m} u_i (\mathbf{x}_i - \mathbf{x}_{i-1}) - u_i \mathbf{x}_i - u_{i-1} \mathbf{x}_{i-1},$$

and thus

$$\mathcal{A}[\nabla u_i](\mathbf{z}) = \sum_{i=1}^{m} \frac{|\tau_i|}{|\omega_2|} \nabla u_i|_{\tau}.$$

The first sum on the right hand side can be bounded by using (22), and the second one can be bounded as follows

$$\left| (\mathbf{x}_i \cdot \nabla)^3 u(\mathbf{z}) (\mathbf{x}_{i+1} - \mathbf{x}_{i-1}) \right| \leq \left| (\mathbf{x}_i \cdot \nabla)^3 u(\mathbf{z}) \right| \cdot \left| (\mathbf{x}_{i-1} + \mathbf{x}_{i-1}) \right|

\leq (|\mathbf{J}_{i+1} \mathbf{\hat{x}}_{i+1}| + |\mathbf{J}_{i-1} \mathbf{\hat{x}}_{i-1}|) \cdot (|\mathbf{J}_{i-1} \mathbf{\hat{x}}_{i-1}|)^3/2

\leq \left( \|\mathbf{J}_{i+1}\| + \|\mathbf{J}_{i-1}\| \right) \cdot \|\mathbf{\hat{J}}_{i} \mathbf{Q}_2 F \|^{3/2},$$

where in the second step $\mathbf{\hat{x}}_i$ is the pre-image of $\mathbf{x}_i$ on the standard element. Finally, we complete the proof of this lemma by noting that

$$\frac{1}{|\omega_2|} \|\mathbf{J}_i\| \leq \frac{1}{|\tau_i|} \|\mathbf{J}_i\| \leq \|\mathbf{J}_{i-1}^{-1}\|. \quad \square$$

**Theorem 3.1.** Suppose the exact solution $u$ of (1) satisfies assumption (7) about its anisotropic behavior, and suppose each pair of adjacent elements in partition $\mathcal{T}_N$ form an $\text{O}(N^{-1/(1+\alpha)})$-approximate parallelogram. Let $\Omega^0_N = \bigcup_{\mathbf{J} \neq 0} \mathbf{J}$ be the region composed of all the interior elements. Then we have

$$\|\mathcal{A}[\nabla u_N] - \nabla u\|_{p_\Omega^0_N} \leq c \max_{\mathbf{J} \in \mathcal{T}_N} \{ (1 + N^{-\alpha/2}) \|\mathbf{J}_i\|^{-2} \|\mathbf{J}_i\| \cdot \|\mathbf{J}^T \mathbf{Q}_2 F\| + \|\mathbf{J}_i^{-1}\|^{-2} \|\mathbf{J}_i\| \cdot \|\mathbf{J}^T \mathbf{Q}_2 F\|^{3/2}. \}

(26)$$

Furthermore, if the partition $\mathcal{T}_N$ is quasi-uniform under metric $M$, then we have

$$\|\mathcal{A}[\nabla u_N] - \nabla u\|_{p_\Omega^0_N} \leq c N^{-1/2} \max_{\mathbf{J} \in \mathcal{T}_N} \{ C_M N^{-1/2} + C_M N^{-\alpha/2} \|F^{-1}\|^{2} \|\mathbf{F}\| \cdot \|\mathbf{F}^T \mathbf{Q}_2 F\| + C_M N^{-1/2} \|F^{-1}\|^{2} \|\mathbf{F}\| \cdot \|\mathbf{F}^T \mathbf{Q}_2 F\|^{3/2}. \}

(27)$$

where $F$ and $C_M$ are determined by metric $M$ as in (4) and (6).

**Proof.** Note that

$$\|\mathcal{A}[\nabla u_N] - \nabla u\|_{p_\Omega^0_N} \leq \|\mathcal{A}[\nabla u_N - \nabla u]\|_{p_\Omega^0_N} + \|\mathcal{A}[\nabla u_i] - \nabla u\|_{p_\Omega^0_N}. \quad (28)$$

Clearly

$$\|\mathcal{A}[\nabla u_N - \nabla u]\|_{p_\Omega^0_N}^{2} \leq c \sum_{\mathbf{J} \in \mathcal{T}_N} \|\nabla u_N - \nabla u_i\|^{2}_{\mathbf{J}} \cdot |\omega_2|$$

$$\leq c \sum_{\mathbf{J} \in \mathcal{T}_N} \|\nabla u_N - \nabla u_i\|^{2}_{\mathbf{J}}.$$
\[ \leq c \| \nabla u_N - \nabla u_I \|^2_{L^2} \]

\[ \leq c \max_{I \in T} \left( 1 + N^{-\alpha/2} \| J_I^{-1} \| \| J_I \| \| \nabla J_I \| + \| J_I^{-1} \| \| \nabla J_I \| \right)^2. \]

where in the last step we have used the superconvergence of \( \nabla u_N - \nabla u_I \) described in Theorem 2.2.

For the second term in (28), let \( \Pi_1 : C(\Omega) \rightarrow S_N \) be the piecewise linear interpolation operator at the vertices of \( T_N \). We have from the triangle inequality that

\[ \| A[\nabla u_I] - \nabla u_I \|_{L^2} \leq \| A[\nabla u_I] - \Pi_1(\nabla u_I) \|_{L^2} + \| \Pi_1(\nabla u_I) - \nabla u_I \|_{L^2}. \]

The first term on the right hand side above can be estimated by Lemma 3.5 and the fact that

\[ \| A[\nabla u_I] - \Pi_1(\nabla u_I) \|_{L^2} \leq |\Omega| \max_{z \in T} \| A[\nabla u_I](z) - \nabla u(z) \|. \]

The second term in (30) can be estimated as in [35,36] for the \( L^2 \)-norm of the linear interpolation error as follows,

\[ \| \Pi_1(\nabla u) - \nabla u \|^2_{L^2} = \sum_{I \in T} \int_I |\Pi_1(\nabla u) - \nabla u|^2 = \sum_{I \in T} |I| \int_I |\Pi_1 \nabla u - \nabla u|^2 \leq c \sum_{I \in T} |I| \int |\nabla_I^2 \nabla u|^2, \]

where \( \nabla_I = J_I^T \nabla \) is the gradient on the standard element. Moreover, let \( e_1 = [1, 0]^T \), it follows from Lemma 3.1 that

\[ \| \nabla_I^2 \partial_y u \| = \sup_{|s| = 1} |(s \cdot \nabla_I^2)(\partial_y u)| \]

\[ = |(J_I s \cdot \nabla)(e_1 \cdot \nabla) u| \]

\[ \leq c \sup_{|s| = 1} |J_I s \cdot Q J_I s| \cdot |e_1 \cdot Q s|^{1/2} \]

\[ \leq c \| J_I^{-1} \| \cdot \| J_I \|^{1/2} \]

\[ \leq c \| J_I^{-1} \| \cdot \| J_I \|^{3/2} \], \]

de and we can bound \( \| \nabla_I^2 \partial_y u \| \) similarly. By putting both of these two estimates into (31), and (31) into (30) in turn, we complete the proof of (26).

Finally, when \( T_N \) is quasi-uniform under metric \( M \), (27) follows easily from (5) and (26). \( \square \)

### 3.2. Recovery based on local \( L^2 \)-projection

We consider here the recovery of \( \nabla u_N \) by \( \tilde{g}_1[\nabla u_N] = \left[ \tilde{g}_N[\partial_y u_N] \right] \), where \( \tilde{g}_1 : L^2(\Omega) \rightarrow S_N \) is defined as follows: for any \( f \in L^2(\Omega) \) and any node \( z \) of \( T_N \), let \( \omega_z = \bigcup_{i=1}^m \tau_i \) be the patch of elements around \( z \), and let \( p_a \) be the \( L^2(\omega_z) \)-projection of \( f \) onto the space of first order polynomials, then we define

\[ \tilde{g}_1[f](z) = p_a(z). \]

Our analysis follows the similar approaches developed by Li and Zhang [33] and Xu and Zhang [34] for the case of quasi-uniform meshes. First, it is obvious that \( \tilde{g}_1[f](z) \) is invariant under any translation of \( \omega_z \) and \( f \) \| \omega_z \|. Thus in the analysis below, we may assume \( z = 0 \). Moreover, it is not difficult to see that \( \tilde{g}_1[f](z) \) is also invariant under a linear transformation of \( \omega_z \). Indeed, let \( \tilde{\mathcal{F}} = \mathcal{F}^{-1}(\omega_z) \) and \( \tilde{f} = f \circ \mathcal{F} \). Then

\[ p_a(\tilde{x}) = \beta_0 + \beta_1 \cdot \tilde{x} \]

is the \( L^2(\omega_z) \) projection of \( f \| \omega_z \), iff the following function is the \( L^2(\omega_z) \)-projection of \( \tilde{f} \)

\[ \tilde{p}_a(\tilde{\mathbf{x}}) = \beta_0 + \mathbf{J} \beta_1 \cdot \tilde{\mathbf{x}}, \]

where \( \mathbf{J} \) is the Jacobian of \( \mathcal{F} \). Thus \( \tilde{g}_1[f](z) = p_a(0) = \tilde{p}_a(0) \).
Lemma 3.6. Suppose each pair of adjacent elements in $T_N$ form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. Then we have for any $v \in L^2(\Omega)$ that

$$\|g_1[\cdot][v]\|_{L^2_N} \leq c\|v\|_{\Omega},$$

where $\Omega_N^0$ is the region composed of all the interior elements.

Proof. We first show that at each node $z \in \Omega_N^0$, we have

$$|g_1[\cdot][v](z)| \leq c|\omega_z|^{-1/2}\|v\|_{\omega_z}.$$ 

In fact, let $J_1$ be the Jacobian of the affine mapping from the standard element $\hat{\cdot}$ to element $\tau_1$ in $\omega_z$, and define $x = \mathcal{F}(\hat{x}) = j_1 \hat{x}$ and $\tilde{v} = v \circ \mathcal{F}$. Since each pair of adjacent elements in $T_N$ form an approximate parallelogram, we have from Lemma 3.3(iii) that $\tilde{\omega}_z = \mathcal{F}^{-1}(\omega_z)$ is shape regular and of diameter $O(1)$. Moreover, $|\omega_z| \leq c|\tau_1|$. Thus

$$|g_1[\cdot][v](z)| = |\tilde{p}_z(0)| \leq c\|\tilde{p}_z\|_{\omega_z} \leq c\|\tilde{v}\|_{\omega_z} \leq c|\det(J_1)|^{-1/2}\|v\|_{\omega_z} \leq c|\omega_z|^{-1/2}\|v\|_{\omega_z}.$$ 

Now let $\phi_z$ be the nodal basis function associated with node $z$, and we have

$$\|g_1[\cdot][v]\|^2_{L^2_N} = \int_{\Omega_N^0} \sum_{z \in \Omega_N^0} g_1[\cdot][v](z)\phi_z(x)^2 \, dx \leq c\sum_{z \in \Omega_N^0} |g_1[\cdot][v](z)|^2 \cdot |\omega_z| \leq c\sum_{z \in \Omega_N^0} \|v\|^2_{\omega_z} \leq c\|v\|^2_{L^2_N}. \square$$

Lemma 3.7. Suppose $z$ is an interior node, and each pair of adjacent elements in patch $\omega_z$ form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. Furthermore, $u$ satisfies assumption (7) about its anisotropic behavior. Then we have

$$|g_1[\nabla u_l](z) - \nabla u(z)| \leq c\max_{\tau \in \omega_z} \|J^{-1}_1\|_{\tau} \|N^{-\alpha/2}\|QJ_T\| + \|J^TQJ_T\|^{3/2}).$$

Proof. Note that

$$|g_1[\nabla u_l](z) - \nabla u(z)| \leq |g_1[\nabla u_l](z) - A[\nabla u_l](z)| + |A[\nabla u_l](z) - \nabla u(z)|,$$

where $A$ is the local averaging operator introduced in the last subsection. The second term on the right hand side above has been dealt with in Lemma 3.5. We only have to bound the first term. For simplicity, we use $\hat{\partial}u$ to represent either $\partial_x u$ or $\partial_y u$. Since both $g_1[\cdot][\partial_q u_l](z)$ and $A[\partial_q u_l](z)$ are invariant under translation, we may assume $z = 0$ without loss of generality.

Let the $L^2(\omega_z)$ projection of $\partial_q u_l|_{\omega_z}$ be

$$p_* = \beta_0 + \beta_1 \cdot x.$$

Clearly

$$\int_{\omega_z} (p_* - \partial_q u_l)q = 0, \quad \forall q \in P_1,$$

which implies

$$A[\partial_q u_l](z) = \frac{1}{|\omega_z|} \int_{\omega_z} \partial_q u_l = \frac{1}{|\omega_z|} \int_{\omega_z} p_*.$$

Thus we have

$$g_1[\partial_q u_l](z) - A[\partial_q u_l](z) = p_*(0) - \frac{1}{|\omega_z|} \int_{\omega_z} p_*$$

and

$$|g_1[\partial_q u_l](z) - A[\partial_q u_l](z)| \leq c\max_{\tau \in \omega_z} \|J^{-1}_1\|_{\tau} \|N^{-\alpha/2}\|QJ_T\| + \|J^TQJ_T\|^{3/2}).$$
Consequently, we have that
\[ \epsilon_i = \left| |\tau_i|/|\tau_1| - 1 \right|. \]

By the assumption that each pair of adjacent elements form an \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram, we have from (10) that \( \epsilon_i = O(N^{-\alpha/2}) \). Thus
\[
\left| \beta_1 \cdot \sum_{i=1}^{m} c_i \cdot |\tau_i| \right| = |\tau_1| \left| \beta_1 \cdot \sum_{i=1}^{m} (c_i + c_i \cdot \epsilon_i) \right| \leq |\tau_1| |J_1 \beta_1| \cdot \left( \left| J_1^{-1} \cdot \sum_{i=1}^{m} c_i \right| + \sum_{i=1}^{m} |J_1^{-1} c_i| \cdot |\epsilon_i| \right),
\]
where \( J_1 \) is the Jacobian of the affine mapping from standard element \( \hat{\tau} \) onto \( \tau_1 \) in \( \omega_2 \). Since \( c_i = \frac{1}{2} (x_{i-1} + x_i + \mathbf{0}) \), it follows from Lemma 3.3(ii) that
\[
\left| J_1^{-1} \cdot \sum_{i=1}^{m} c_i \right| = \frac{2}{3} \left| J_1^{-1} \cdot \sum_{i=1}^{m} x_i \right| \leq cN^{-\alpha/2},
\]
and from Lemma 3.3(i) that
\[
\sum_{i=1}^{m} |J_1^{-1} c_i| \cdot |\epsilon_i| \leq cN^{-\alpha/2} \sum_{i=1}^{m} |J_1^{-1} c_i| \leq cN^{-\alpha/2} \sum_{i=1}^{m} ||J_1^{-1}|| \cdot |J_1^{-1} c_i| \leq cN^{-\alpha/2}.
\]

Consequently, we have that
\[
|\hat{g}_{1}[\partial u_1](\mathbf{z}) - \mathcal{A}[\partial u_1](\mathbf{z})| \leq cN^{-\alpha/2} |J_1 \beta_1|. \tag{37}
\]

Next we estimate \( |J_1 \beta_1| \). Define a linear mapping
\[
\mathbf{x} = \mathcal{F}(\xi) = J_1 \xi.
\]
Let \( \hat{\omega}_2 = \mathcal{F}^{-1}(\omega_2) \), \( \hat{\partial u} = \partial u \circ \mathcal{F} \), and let \( \hat{\mathcal{P}}_s \) be the \( L^2(\hat{\omega}_2) \)-projection of \( \hat{\partial u} \). By (33) we know that \( J_1 \beta_1 = \nabla_\xi \hat{\mathcal{P}}_s \). To derive a bound for it, let \( u(\mathbf{x}) \) be the exact solution of (1), and \( \partial u = \partial u \circ \mathcal{F} \). Define \( L(\xi) \) be the first order Taylor’s polynomial of \( \partial u \) at \( \mathbf{0} \), i.e.,
\[
L(\xi) = \hat{\partial u}(\mathbf{0}) + \nabla_\xi \hat{\partial u}(\mathbf{0}) \cdot \xi.
\]

By the triangle inequality
\[
|J_1 \beta_1| = |\nabla_\xi \hat{\mathcal{P}}_s| \leq |\nabla_\xi (\hat{\mathcal{P}}_s - L)| + |\nabla_\xi L|
\]
\[
= |\nabla_\xi (\hat{\mathcal{P}}_s - L)| + |\nabla_\xi \hat{\partial u}(\mathbf{0})|. \tag{38}
\]

Note that \( \nabla_\xi = J^T \nabla \), and \( \partial u = \mathbf{e} \cdot \nabla u \) where \( \mathbf{e} \) is either \([1, 0]^T\) or \([0, 1]^T\), we have
\[
|\nabla_\xi \hat{\partial u}(\mathbf{0})| = \sup_{|s|=1} |s \cdot \nabla_\xi \hat{\partial u}(\mathbf{0})|
\]
\[
= \sup_{|s|=1} |(J_1 s \cdot \nabla) \hat{\partial u}(\mathbf{z})| = \sup_{|s|=1} |(J_1 s \cdot \nabla)(\mathbf{e} \cdot \nabla) u(\mathbf{z})|.
\]

By using Lemma 3.1 and assumption (7),
\[
|\nabla_\xi \hat{\partial u}(\mathbf{0})| = \sup_{|s|=1} |(J_1 s \cdot Q_s(\mathbf{z}) J_1 s)|^{1/2} (|\mathbf{e} \cdot Q_s(\mathbf{z}) \mathbf{e}|)^{1/2}
\]
\[
\leq ||J_1^T Q_s(\mathbf{z}) J_1||^{1/2} ||Q_s(\mathbf{z})||^{1/2}
\]
\[
\leq ||J_1^{-1}|| ||J_1^T Q_s(\mathbf{z}) J_1||. \tag{39}
\]
To estimate $|\nabla \xi (\hat{p}_* - L)|$, we note that $\hat{a}_2$ is of diameter $O(1)$. Therefore

$$|\nabla \xi (\hat{p}_* - L)| \leq \|\hat{p}_* - L\|_{\hat{a}_2}. \tag{40}$$

Furthermore, by the definition of $\hat{p}_*$,

$$\|\hat{p}_* - L\|_{\hat{a}_2} \leq \|\hat{p}_* - \hat{d}_1\|_{\hat{a}_2} + \|\hat{d}_1 - L\|_{\hat{a}_2}$$

$$\leq 2\|\hat{d}_1 - L\|_{\hat{a}_2}$$

$$\leq 2(\|\hat{d}_1 - \hat{d}_1\|_{\hat{a}_2} + \|\hat{d}_1 - L\|_{\hat{a}_2} \tag{41}).$$

By using the anisotropic error estimate for linear interpolations (see [35,36]), we have

$$\|\hat{d}_1 - \hat{d}_1\|_{\hat{a}_2}^2 = \int_{\hat{a}_2} |\hat{d}_1 - \hat{d}_1|^2$$

$$= |\det(J_1)|^{-1} \int_{\hat{a}_2} |\hat{d}_1 - \hat{d}_1|^2$$

$$= \frac{|\hat{r}|}{|\tau_1|} \sum_{\tau_i=1}^m \int_{\tau_i} |\hat{d}_1 - \hat{d}_1|^2$$

$$\leq \frac{|\hat{r}|}{|\tau_1|} \sum_{\tau_i=1}^m \|\hat{d}_1 - \hat{d}_1\|_{\tau_i}^2$$

$$= c \max_{\tau \in \hat{a}_2} \|\hat{d}_1 - \hat{d}_1\|_{\tau_i}^2 \tag{42}.$$

Similarly to the estimate (32), we have

$$\|\hat{d}_1 - L\|_{\hat{a}_2} \leq c \max_{\hat{a}_2} \|\nabla \hat{d} \hat{u}(\xi)\|$$

$$\leq c \max_{\tau \in \hat{a}_2} \|\hat{d}_1 - \hat{d}_1\|_{\tau_i} \|\hat{d}_1 - L\|_{\hat{a}_2} \tag{43}.$$

Putting the above two estimates into (41) and then into (38), we have

$$|J_1 \beta_1| \leq c \max_{\tau \in \hat{a}_2} \|\hat{d}_1 - \hat{d}_1\|_{\tau_i} \|\hat{d}_1 - L\|_{\hat{a}_2} \tag{44}.$$

The proof is completed by putting the above estimate into (37). \qed

**Theorem 3.2.** Suppose the exact solution $u$ of (1) satisfies assumption (7) about its anisotropic behavior, and suppose each pair of adjacent elements in partition $\mathcal{T}_N$ form an $O(N^{-(1+\alpha)})$-approximate parallelogram. Then we have over the region $\Omega_0^N$ composed of all the interior elements that

$$\|\tilde{g}_i \nabla u_N\| - \nabla u_N\|_{\Omega_0^N} \leq c \max_{\tau \in \mathcal{T}_N} (1 + N^{-\alpha/2} \|J_1^{-1}\| \|J_1\| \cdot \|J_1 T Q J_1\| + \|J_1^{-1}\| \|J_1\| \cdot \|J_1 T Q J_1\|^{3/2}) \tag{45}.$$

Furthermore, if the partition $\mathcal{T}_N$ is quasi-uniform under metric $M$, then we have

$$\|\tilde{g}_i \nabla u_N\| - \nabla u_N\|_{\Omega_0^N} \leq c N^{-1/2} \max_{\tau \in \mathcal{T}_N} (C_M N^{-1/2} + C_M N^{-\alpha/2} \|F^{-1}\| \|F\| \cdot \|F T Q F\|$$

$$+ C_M N^{-1/2} \|F^{-1}\| \|F\| \cdot \|F T Q F\|^{3/2}) \tag{46}.$$

where $F$ and $C_M$ are determined by metric $M$ as in (4) and (6).

**Proof.** By the triangle inequality,

$$\|g_i \nabla u_N\| - \nabla u_N\|_{\Omega_0^N} \leq \|g_i \nabla u_N\| - \|g_i \nabla u_N\|_{\Omega_0^N} + \|g_i \nabla u_N\| - \|g_i \nabla u_N\|_{\Omega_0^N} + \|g_i \nabla u_N\| - \|g_i \nabla u_N\|_{\Omega_0^N} \tag{47}.$$

Note that the first term on the right hand side above can be estimated by the superconvergence of $\|\nabla (u_N - u_i)\|$ (Theorem 2.2) and the boundedness of operator $g_i$ (Lemma 3.6), while the third term has been dealt with in the proof of Theorem 3.1. The second term can be bounded by (37) and the fact that

$$\|g_i \nabla u_N\| - \|g_i \nabla u_N\|_{\Omega_0^N} \leq |\Omega| \max_{z \in \mathcal{O}_N^0} \|g_i \nabla u_N(z) - A[\nabla u_N](z)\|.$$

The second part of this theorem follows from (45) and (5). \qed
3.3. Recovery based on least square fitting

Consider a recovery of \( \nabla u_N \) by \( \hat{g}_2[\nabla u_N] = \left[ \hat{g}_2[\partial_1 u_N] \right] \), where \( \hat{g}_2 \) is defined as follows: for any node \( z \) of \( T_N \), let \( \omega_z = \bigcup_{i=1}^m \tau_i \) be the patch of elements around \( z \), and \( \{ \mathbf{c}_i \}_{i=1}^m \) are the centers of \( \{ \tau_i \}_{i=1}^m \). Let \( p_* \in P_1 \) be the least square fitting of the data \( \{(\mathbf{c}_i, f(\mathbf{c}_i))\}_{i=1}^m \), i.e., \( p_* \) is the minimizer of

\[
\sum_{i=1}^m |p_1(\mathbf{c}_i) - f(\mathbf{c}_i)|^2
\]

for all first order polynomials \( p_1 \). Then we define

\[
\hat{g}_2[f](z) = p_*(z).
\]

\( \hat{g}_2 \) has similar properties as \( \hat{g}_1 \), e.g., \( \hat{g}_1[f](z) \) is also invariant under any translation of \( \omega_z \) and \( f |_{\omega_z} \). Also if \( x = \mathcal{F}(\hat{x}) \) is a linear transform, and \( \hat{\omega}_z = \mathcal{F}^{-1}(\omega_z) \) and \( \hat{f} = \mathcal{F} \circ f \), then the least square fitting of \( \hat{f} \) on \( \hat{\omega}_z \) is

\[
\hat{p}_*(\hat{x}) = \beta_0 + J \beta_1 \cdot \hat{x},
\]

where \( p_*(x) = \beta_0 + \beta_1 \cdot x \) is the least square fitting for \( f \) on \( \omega_z \), and \( J \) is the Jacobian of \( \mathcal{F} \).

Lemma 3.8. Suppose \( \tau_1 \cup \tau_2 \) forms an \( O(N^{-(1+\alpha)/2}) \)-approximate parallelogram, and \( m \) is the midpoint of the common side \( \tau_1 \cap \tau_2 \). Then for any function \( u \) satisfying (7), we have

\[
\frac{1}{2} (\nabla u |_{\tau_1} + \nabla u |_{\tau_2}) - \nabla u (m) \leq c \max_{1 \leq i \leq 2} \| \nabla u |_{\tau_i} \|_i (N^{-1/2} \| Q_j f_j \| + \| J_j Q_j f_j \|^{3/2}).
\]

Proof. Assume without loss of generality that \( \tau_1 = \Delta x_1 x_2 x_3, \tau_2 = \Delta x_1 x_3 x_4, \) and \( m = \frac{1}{2}(x_1 + x_3) = 0 \). By a formula similar to (19) we have

\[
\nabla u |_{\tau_1} = -\frac{1}{2|\tau_1|} R_{\frac{1}{2}} \{ u_1(x_2 - x_3) + u_2(x_3 - x_1) + u_3(x_1 - x_2) \}.
\]

By using Taylor’s expansion for \( u(x) \) at \( m \), we have

\[
u_i = u(x_i) = u(m) + (x_i \cdot \nabla) u(m) + \frac{1}{2} (x_i \cdot \nabla)^2 u(m) + \frac{1}{6} (x_i \cdot \nabla)^3 u(m_i),
\]

where \( m_i \) is in between \( m \) and \( x_i \). Put the above equation into its previous equation and note that \( x_1 = -x_3 \), we have

\[
\nabla u |_{\tau_1} = \nabla u (m) - \frac{1}{2|\tau_1|} R_{\frac{1}{2}} \{ x_i[(x_1 \cdot \nabla)^2 u(m) - (x_2 \cdot \nabla)^2 u(m)] + \sum_{i=1}^3 \varepsilon_i \}
\]

where

\[
|\varepsilon_i| = \frac{1}{6|\tau_1|} |(x_i - x_1 - x_1)(x_i \cdot \nabla)^3 u(m_i)|
\]

\[
\leq \frac{1}{6|\tau_1|} |(x_i - x_1)| |(x_i + x_1)| |x_i \cdot Q_j x_i|^{3/2}
\]

\[
\leq \frac{c}{|\tau_1|} \| J_j \| \| Q_j f_j \|^{3/2}
\]

Similarly we have for \( \nabla u |_{\tau_2} \) that

\[
\nabla u |_{\tau_2} = \nabla u (m) - \frac{1}{2|\tau_2|} R_{\frac{1}{2}} \{ -x_1[(x_4 \cdot \nabla)^2 u(m) - (x_3 \cdot \nabla)^2 u(m)] + \sum_{i=4}^6 \varepsilon_i \}
\]

where for \( 4 \leq i \leq 6 \),

\[
|\varepsilon_i| \leq c \| J_j^{-1} \| \| Q_j f_j \|^{3/2}.
\]
Combining the above two equations, we have
\[
\left| \frac{1}{2} (\nabla u_{|r_1} + \nabla u_{|r_2}) - \nabla u(m) \right| \leq \frac{1}{2} \left( \frac{x_1}{|r_1|} - \frac{x_1}{|r_2|} \right) (x_1 \cdot \nabla)^2 u(m) + \frac{x_1}{|r_1|} (x_4 \cdot \nabla)^2 \nabla u(m) - \nabla u(m) + \sum_{i=1}^{6} |E_i| \\
\leq \frac{1}{2} \left( \frac{x_1}{|r_1|} - \frac{x_1}{|r_2|} \right) ((x_1 \cdot \nabla)^2 u(m) + (x_4 \cdot \nabla)^2 u(m)) \right| + \frac{1}{2} \left[ (x_4 \cdot \nabla)^2 u(m) - (x_2 \cdot \nabla)^2 u(m) \right] + \sum_{i=1}^{6} |E_i|. \tag{50}
\]

By the facts that
\[
\left| \frac{1}{|r_1|} - \frac{1}{|r_2|} \right| = \frac{1}{|r_1|} \cdot \frac{1}{|r_2|} - 1 \leq \frac{c}{|r_1|} N^{-\alpha/2},
\]
and
\[
|J^{-1}(x_2 + x_4)| \leq cN^{-\alpha/2}, \quad i = 1, 2,
\]
we can bound the first two terms on the right hand side of (50) by
\[
c N^{-\alpha/2} \max_{i=1,2} \left| \frac{J_i}{|r_1|} J_i Q J_i \right|.
\]
Finally by noticing that \( |J_i/|/|r_1| \leq c |J_i^{-1}| \), we reach the conclusion of this lemma. \(\square\)

**Lemma 3.9.** Suppose \( z \) is an interior node, and each pair of adjacent elements in patch \( \omega_z \) form an \( O(N^{-\alpha/2}) \)-approximate parallelogram. Furthermore, \( u \) satisfies assumption (7) about its anisotropic behavior. Then for the simple average of \( \nabla u_i \) over \( \omega_z \)
\[
\tilde{A}[\nabla u_i](z) = \frac{1}{m} \sum_{i=1}^{m} \nabla u_i_{|r_i},
\]
we have
\[
|\tilde{A}[\nabla u_i](z) - \nabla u(z)| \leq c \max_{r \in \omega_z} \left| \frac{J_i^{-1}}{1} \right| (N^{-\alpha/2} |J_i T Q J_i| + |J_i T Q J_i|^{3/2}). \tag{52}
\]

**Proof.** Assume without loss of generality that \( z = 0 \). Then
\[
|\tilde{A}[\nabla u_i](z) - \nabla u(z)| \leq \frac{1}{m} \sum_{i=1}^{m} \frac{1}{2} (\nabla u_{|r_i} + \nabla u_{|r_i+1}) - \nabla u(m_i) \right| + \frac{1}{m} \sum_{i=1}^{m} \nabla u(m_i) - \nabla u(z),
\]
where \( m_i = \frac{1}{2} x_i \) is the midpoint between \( x_i \) and \( z \). The first term on the right hand side can be dealt with by Lemma 3.8. The second term can be easily estimated as in (24) by using Taylor’s expansion for \( \nabla u \) at \( z \) and the fact that \( |J_i^{-1} \sum_{i=1}^{m} m_i| = O(N^{-\alpha/2}) \). \(\square\)

**Lemma 3.10.** Suppose \( z \) is an interior node, and each pair of adjacent elements in patch \( \omega_z \) form an \( O(N^{-\alpha/2}) \)-approximate parallelogram. Furthermore, \( u \) satisfies assumption (7) about its anisotropic behavior. Then we have
\[
|g_2[\nabla u_i](z) - \nabla u(z)| \leq c \max_{r \in \omega_z} \left| \frac{J_i^{-1}}{1} \right| (N^{-\alpha/2} |J_i T Q J_i| + |J_i T Q J_i|^{3/2}). \tag{53}
\]

**Proof.** This lemma can be proved in a similar way as in the proof of Lemma 3.7. Note that
\[
|g_2[\nabla u_i](z) - \nabla u(z)| \leq |g_2[\nabla u_i](z) - \tilde{A}[\nabla u_i](z)| + |\tilde{A}[\nabla u_i](z) - \nabla u(z)|.
\]
where \( \tilde{A} \) is the simple averaging operator introduced in (51). The second term on the right hand side above has been dealt with in Lemma 3.9. We only have to estimate the difference \( g_2[\nabla u_i](z) - \tilde{A}[\nabla u_i](z) \) at node \( z \). Since both \( g_2[\nabla u_i](z) \) and \( \tilde{A}[\nabla u_i](z) \) are invariant under translation, we may assume \( z = 0 \). Let \( p_\beta = \beta_0 + \beta_1 \cdot x \) be the linear polynomial determined by least square fitting of \( \nabla u_i \) over the element centers \( \{c_i\}_{i=1}^{m} \) in \( \omega_z \). Clearly, we have
\[
\sum_{i=1}^{m} p_\beta(c_i) = \sum_{i=1}^{m} \nabla u_{|r_i} = m \tilde{A}[\nabla u_i](z).
\]
Thus
\[
|g_2[\nabla u_I](\mathbf{x}) - \bar{A}[\nabla u_I](\mathbf{x})| = \left| p_*(0) - \frac{1}{m} \sum_{i=1}^{m} p_*(\mathbf{c}_i) \right|
\]
\[
= \left| \frac{1}{m} \sum_{i=1}^{m} \beta_1 \cdot \mathbf{c}_i \right|
\]
\[
\leq \frac{1}{m} |j_1 \beta_1| \cdot \left| j_1^{-1} \sum_{i=1}^{m} \mathbf{c}_i \right|
\]
\[
\leq c N^{-\alpha/2} |j_1^{-1} \beta_1|.
\]
The estimate of $|j_1 \beta_1|$ follows in the same way as in the proof of (37) in Lemma 3.7, except instead of (40) and (41) we use
\[
|\nabla \hat{p}_* - L| \leq c \sum_{i=1}^{m} |\hat{p}_*(\mathbf{c}_i) - L(\mathbf{c}_i)|
\]
\[
\leq c \cdot 2 \sum_{i=1}^{m} |\hat{u}_i(\mathbf{c}_i) - L(\mathbf{c}_i)|
\]
\[
\leq c \sum_{i=1}^{m} (|\hat{u}_i(\mathbf{c}_i) - \bar{u}(\mathbf{c}_i)| + |\bar{u}(\mathbf{c}_i) - L(\mathbf{c}_i)|).
\]

Each of the two terms on the right hand sides above can be bounded as in (42) and (43). This concludes our proof of this lemma. □

Theorem 3.3. Suppose the exact solution $u$ of (1) satisfies assumption (7) about its anisotropic behavior, and suppose each pair of adjacent elements in partition $\mathcal{T}_N$ form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram. Then we have over the region $\Omega^0_N$ composed of all the interior elements that
\[
\|g_2[\nabla u_N] - \nabla u\|_{L^2(N)} \leq c \max_{\mathcal{T} \in \mathcal{T}_N}\{1 + N^{-\alpha/2} ||J^{-1}||^2 ||F|| \cdot ||J^{-1}Q_J|| + ||J^{-1}||^2 ||J|| \cdot ||J^T Q_J J||^{3/2} \}.
\]

Furthermore, if the partition $\mathcal{T}_N$ is quasi-uniform under metric $M$, then we have
\[
\|g_2[\nabla u_N] - \nabla u\|_{L^2(N)} \leq c N^{-1/2} \max_{\mathcal{T} \in \mathcal{T}_N}\{C_M N^{-1/2} + C_M N^{-\alpha/2} ||F||^2 ||F|| \cdot ||F^T Q_J F|| + C_M N^{-1/2} ||F||^2 ||F|| \cdot ||F^T Q_J F||^{3/2} \},
\]
where $F$ and $C_M$ are determined by metric $M$ as in (4) and (6).

Proof. Let $\bar{A}[\nabla u_I] \in S_N$ be determined by its nodal value as in (51). Then we have
\[
\|g_2[\nabla u_N] - \nabla u\|_{L^2(N)} \leq \|g_2[\nabla u_N] - g_2[\nabla u_I]\|_{L^2(N)} + \|g_2[\nabla u_I] - \bar{A}[\nabla u_I]\|_{L^2(N)} + \|\bar{A}[\nabla u_I] - \nabla u\|_{L^2(N)}.
\]

Note that similar to $g_1$ in Lemma 3.6, $g_2$ is bounded. Then with the help of Lemmas 3.9 and 3.10 this theorem can be proved in the same way as for Theorem 3.2. □

Remark 3.2. It is easy to see from the proof of Theorem 3.3 that a gradient recovery $\bar{A}(\nabla u_N)$ based on simple averaging also enjoys the same superconvergence property as the weighted averaging $A(\nabla u_N)$ does.

Remark 3.3. Based on a gradient recovery $\mathcal{R}[\nabla u_N]$, where $\mathcal{R}$ can be any one of the local weighted averaging $A$, local $L^2$-projection $g_1$, or least square fitting $g_2$, we may define an a-posteriori error estimator $\|\nabla u_N - \mathcal{R}[\nabla u_N]\|$. Under the conditions that $\mathcal{T}_N$ is quasi-uniform under metric $M$ and each pair of adjacent elements form an $O(N^{-(1+\alpha)/2})$-approximate parallelogram, we have the asymptotic exactness of the error estimator, i.e.,
\[
\left| \frac{\|\nabla u_N - \mathcal{R}[\nabla u_N]\|_{L^2(N)}}{\|\nabla u_N - \nabla u\|_{L^2(N)}} - 1 \right| = O(N^{-\alpha/2}),
\]
provided the true error $\|\nabla u_N - \nabla u\|_{L^2(N)}$ decays at the rate of $N^{-1/2}$. 
4. Numerical results

We present in this section some numerical results for the three types of recovery techniques analyzed in the last section. We consider the Poisson problem (1) in domain \( \Omega = [0, 1]^2 \) with the right hand side function \( f \) and the Dirichlet boundary condition being selected so that it admits the following exact solution:

\[
u(x, y) = 4[1 - e^{-Kx} - x(1 - e^{-K})] \cdot [y(1 - y)],
\]

where \( K = 100 \). This example involves a solution with steep front along the left boundary of the domain, see Fig. 2(a). It has been used in [6, 21, 23, 24] for the numerical study of some a-posteriori error estimates on anisotropic meshes.

Clearly, an ideal mesh for the finite element approximation of this problem should have long and thin elements aligned with the \( y \)-axis near the left side boundary. To obtain such adaptive meshes, we define a mesh metric based on the \( H^1 \) error estimate for linear interpolation as follows:

\[
M = c[\lambda_{\text{max}}/\lambda_{\text{min}}]^{1/4} Q_2,
\]

(58)

where \( Q_2 = \text{abs}(\nabla^2 u) + 1 \) characterizes the anisotropic behavior of \( \nabla^2 u \). The identity matrix \( I \) is added here to avoid the metric \( M \) becoming degenerate in case \( \nabla^2 u \) does so. \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the largest and smallest eigenvalues of \( Q_2 \), respectively. \( c \) is a positive constant to control the total number of elements in the mesh. This particular form of metric was derived from the minimization of an upper bound of the \( H^1 \)-seminorm of the linear interpolation errors in our early work, see [35, 36, 41] for details. Based on this metric, we use a two-dimensional anisotropic generator [8], bamg, to create the adaptive meshes of various sizes. A typical mesh of 4126 elements (generated with \( c = 90 \) in (58)) is displayed in Fig. 2. Its highest element aspect ratio is about 100. Clearly, the meshes generated from bamg do not satisfy that each pair of adjacent elements forms an approximate parallelogram. A more appropriate setting for this type of meshes might be the so-called \((\alpha, \sigma)\) condition introduced in Xu and Zhang [34], where the majority element pairs form \( O(h^{-1+\omega}) \) approximate parallelograms, while the rest are of measure \( O(h^{-\sigma}) \) only. Our analysis can be extended to this type of meshes with some modification. We choose not to do so here to keep our presentation simple and clear. However we present some statistics indicating that, as the meshes get finer and finer, more and more element pairs in the adaptive meshes become closer and closer to parallelograms. To be more specific, we recall the following measure of “closeness” to a parallelogram introduced in [30] for a pair of adjacent elements \( \tau_1 \) and \( \tau_2 \) sharing an edge \( e \):

\[
\mathcal{M}_\alpha(e) = \frac{1}{2} (\| J_1 + J_2 \|^2 + \| J_2 - J_1 \|^2),
\]

(59)

where \( J_1 \) and \( J_2 \) are the Jacobian of the affine mapping from an equilateral standard element \( \hat{\tau} \) to \( \tau_1 \) and \( \tau_2 \), respectively, with the two vertices opposite to \( e \) being the images of the same vertex of \( \hat{\tau} \). The matrix norm in (59) is chosen to be Frobenius norm for easy calculation. Note that when \( \tau_1 \) and \( \tau_2 \) form an exact parallelogram, \( \mathcal{M}_\alpha(e) = 0 \). When \( \tau_1 \) and \( \tau_2 \) form an \( O(N^{-1+\omega}/2) \) approximate parallelogram, \( \mathcal{M}_\alpha(e) = O(N^{-\omega/2}) \). If both \( \tau_1 \) and \( \tau_2 \) are shape regular, then \( \mathcal{M}_\alpha(e) = O(1) \), while if \( \tau_1 \) is shape regular but \( \tau_2 \) is of a high aspect ratio, then \( \mathcal{M}_\alpha \gg 1 \).

We display in Fig. 3(a) the percentile of the element pairs whose \( \mathcal{M}_\alpha \) are below a given value specified by the horizontal axis, and in Fig. 3(b) the average value of \( \mathcal{M}_\alpha \) over all edges for each of the adaptive meshes reported in Tables 1–3. It is clear from these two figures that as the mesh gets refined by bamg, \( \mathcal{M}_\alpha \) decreases in general, and more and more element pairs become closer and closer to exact parallelograms.

We examine the following errors involved at various stages of our analysis:

\[
E_{u-u_N} = \| \nabla u - \nabla u_N \|, \quad E_{u-R_{y}} = \| \nabla u - R(\nabla u) \|, \quad E_{u-R_{u}} = \| \nabla u - R(\nabla u_N) \|.
\]

Here \( R \) represent one of the three recovery techniques analyzed in the last section: \( R = A \) for the recovery based on local weighted averaging, \( R = \hat{g}_1 \) for the local \( L^2 \)-projection, and \( R = \hat{g}_2 \) for the least square fitting (ZZ-type recovery). We also consider the above error norms over the domain \( \Omega_N^0 \) composed of all the interior elements. The errors over \( \Omega_N^0 \) are indicated by a superscript 0, e.g., \( E_{u-R_{y}}^0 \) is the same as \( E_{u-R_{y}} \) except over \( \Omega_N^0 \) instead of the entire \( \Omega \). In addition, we check the effectivity factor \( \eta \) of the error estimator \( E_{u-R_{y}} \):

\[
\eta = E_{u-R_{y}}/E_{u-u_N}.
\]

We list in three tables the various errors for the gradient recoveries based on the local weighted averaging, local \( L^2 \)-projection, and least square fitting, separately. We perform a linear regression of \( \log(\eta) \) vs. \( \log(N) \) to find the rate of convergence \( \beta \) for each type of error \( E \), i.e.,

\[
E \approx cN^{-\beta/2}.
\]

It is easy to note that the convergence of the finite element solution, i.e., \( E_{u-u_N} \), is about linear (with respect to \( N^{-1/2} \)). The convergence of the recovered gradients for both the interpolation of exact solution, \( u_t \), and the finite element solution, \( u_N \), are super-linear \( O(N^{-\beta/2}) \), with \( \beta \) ranging from 1.48 to 1.81. The errors calculated for only interior elements behave similarly to those over the entire domain.
For all three types of recovery techniques, the error estimates are asymptotically exact, with the effectivity factor $\eta$ converges to 1 at approximately rate $O(N^{-\beta/2})$, with $\beta$ ranging from 1.83 to 2.58. In particular, the recovery based on least square fitting produces the highest rate of convergence. We also noticed that when $N$ is large, over 60,000 in our example here, the convergence of $\eta$ fluctuates a little for the recoveries based on local weighted averaging and local $L^2$-projection. We believe this is caused by the observation that for adaptive meshes with large $N$, the tendency to have more element pairs closer to exact parallelograms slows down, refer to Fig. 3. Since the mesh generator bamg we used here is designed mainly to satisfy the mesh metric requirement, it is unclear to us whether this tendency is true in general.

5. Conclusion and discussions

We presented in this paper an analysis on the superconvergence patch recovery techniques for the linear finite element approximation based on adaptive anisotropic meshes. Under the conditions that the meshes are quasi-uniform under a Riemannian metric and that each pair of adjacent elements form an approximate parallelogram, we proved the superconvergence for three types of gradient recovery techniques: the recovery based on local weighted averaging, the recovery based on local $L^2$-projection, and the recovery based on local least square fitting (the well-known ZZ-type recovery).
Further studies are clearly needed in this direction. Applications. A different possible avenue to enhance the superconvergence of the recovery techniques on unstructured PDEs and reliable assessment of the solution errors. Additional work is needed on how to generate such meshes for general gradient recovery and a-posteriori error estimates. A combination of the two is required for the efficient solution of the PDEs and reliable assessment of the solution errors. Additional work is needed on how to generate such meshes for general applications. A different possible avenue to enhance the superconvergence of the recovery techniques on unstructured meshes is to add certain smoothing steps in the post-processing as described in [40] by Bank and Xu for the recovery based on global $L^2$-projection. However, it is unclear how such an idea can be applied to the locally based patch recovery techniques. Further studies are clearly needed in this direction.

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References


