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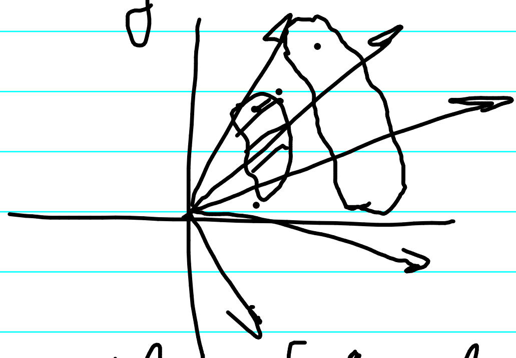
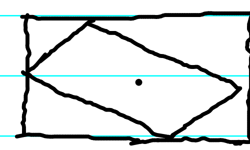
ODE II Jan 23

Recall the "t-advance mapping" g^t of the linear equation $\dot{x} = Ax$ multiplies the volume of any "figure" by $e^{t \operatorname{Tr} A}$

$$\det g^t = \det e^{tA} = e^{\operatorname{Tr} tA} = e^{t \operatorname{Tr} A}$$





$$e^{tA} \qquad \{ \dot{x} = Ax, \quad x(0) = x_0 \}$$

$$x(t) = e^{tA} x_0$$

$$\dot{x}_1(t) = a\, x_1(t) + b\, x_2(t)$$
$$\dot{x}_2(t) = c\, x_1(t) + d\, x_2(t)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$y'' + y = 0 \qquad y(0) = \alpha$$
$$\lambda^2 + 1 = 0 \qquad y'(0) = \beta$$

$$0 = \lambda^2 + 1 = \lambda^2 - i^2 = (\lambda + i)(\lambda - i)$$

$\lambda = \pm i$ $\quad y_1(t) = e^{it} \quad y_2(t) = e^{-it}$

$$y(t) = c_1 e^{it} + c_2 e^{-it}$$

or $y(t) = c_1 \sin(t) + c_2 \cos(t)$

Sec 18 Complex Eigenvalues

$\mathbb{C}^n$ n-dim linear space over $\mathbb{C}$

Decomplexification of $\mathbb{C}^n$ is real linear space which coincides with $\mathbb{C}^n$ as a group, and where mult. by reals is defined as before, mult. by non-real complex nos. is undefined.

Decomplex of $\mathbb{C}^n$ is $2n$-dim space $\mathbb{R}^{2n}$.

$\{e_1, \dots, e_n\}$ basis in $\mathbb{C}^n$, then

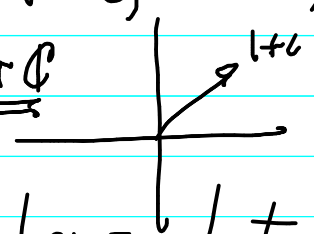
$\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$ forms a basis for ${}^{\mathbb{R}}\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

<u>Decomplexification</u> of $\mathbb{C}$-linear operator

$A: \mathbb{C}^m \to \mathbb{C}^n$ is $\mathbb{R}$-linear operator

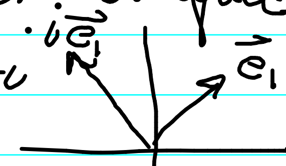
${}^{\mathbb{R}}A: {}^{\mathbb{R}}\mathbb{C}^m \to {}^{\mathbb{R}}\mathbb{C}^n$ which coincides with A point wise. Example:

$\mathbb{C}^1$ $\{\vec{e}_1 = 1+i\}$ is a basis for $\mathbb{C}^1$, a 1-dim vector space <u>over $\mathbb{C}$</u>



${}^{\mathbb{R}}\mathbb{C}^1$ vectors are complex numbers, but scalar field is $\mathbb{R}$ <u>not</u> $\mathbb{C}$

$S = \{t\vec{e}_1 ; t \in \mathbb{R}\}$ is a 1-dim subspace of ${}^{\mathbb{R}}\mathbb{C}^1$. $i\vec{e}_1 = i(1+i) = -1+i$ is linearly independent of $\vec{e}_1$ and $\{\vec{e}_1, i\vec{e}_1\}$ forms a basis for ${}^{\mathbb{R}}\mathbb{C}^1$.



Suppose $\vec{e}_1, \dots, \vec{e}_n$ basis for $\mathbb{C}^n$

$\text{span}\{\vec{e}_1, \dots, \vec{e}_n\} = \mathbb{C}^n$ scalars in $\mathbb{C}$!

$\sum_j c_j \vec{e}_j = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$

Let $\vec{x} = \sum_j c_j \vec{e}_j$ in $\mathbb{C}^n$

$= \sum_j (a_j + i b_j) \vec{e}_j \qquad a_j, b_j \in \mathbb{R}$

$= \sum_j a_j \vec{e}_j + \sum_j b_j \, i\vec{e}_j$

a linear comb. of $\vec{e}_1, \dots \vec{e}_n, i\vec{e}_1, \dots, i\vec{e}_n$ over $\mathbb{R}$.

Suppose $0 = \sum a_j \vec{e}_j + \sum b_j \, i\vec{e}_j$

$= \sum_j (a_j + i b_j) \vec{e}_j \qquad a_j, b_j \in \mathbb{R}$

$\Rightarrow a_j + i b_j = 0 \Rightarrow a_j = b_j = 0$

$1 \le j \le n$.

(A) matrix of A $\mathbb{C}$-linear

$(?)$ matrix of $\mathbb{R}A$ $\mathbb{R}$-linear

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$$|A| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

$$= \inf \{ M : \text{for all } x, \ \|Ax\| \leq M\|x\| \}$$

Show $|A+B| \leq |A| + |B|$

$$|A+B| = \sup \{ \|(A+B)(x)\| : \|x\| = 1 \}$$

$$\|(A+B)(x)\| = \|Ax + Bx\|$$

$$\leq \|Ax\| + \|Bx\|$$

$$|A| = \sup \{ \|Ax\| : \|x\| = 1 \} = S_A$$

$$|B| = \sup \{ \|Bx\| : \|x\| = 1 \}$$