Functions determined by successive approximations.

Theorem. If \( \mathbb{I} \), \( \mathbb{J} \) is an ordered number pair or point, then there exists only one function \( f \) with initial set the set of all numbers such that \( f \) contains \( \mathbb{I} \), \( \mathbb{J} \) and

\[ f' = f, \]

\[ \mathbb{J}, f' = f, \]

\[ \mathbb{I}, f(a) = b \]

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Lemmas:
1. If \( \mathbb{I} \), \( \mathbb{J} \) is a point and \( f \) a function with initial set the set of all numbers, the following are equivalent:
   (i) \( f(a) = b \) and \( f' = f \).
(ii) If \( x \) is a number, then
\[
f(x) = b + \int_a^x f.
\]

2. Suppose \( f \) is a function with initial set the set of all numbers continuous at each of its points, and for each positive integer \( n \) and number \( x \),
\[
f_n(x) = b + \int_a^x f_{n-1}.
\]

Then, \( f_n \) is continuous at each of its points and
\[
f_{n+1}(x) - f_n(x) = \int_a^x (f_n - f_{n-1}).
\]

3. If \([A, B]\) is an interval containing the number \( a \), there exists a number \( M \) such that, if \( x \) is in \([A, B]\), then
\[
|f_1(x) - f_0(x)| \leq M
\]
and for each positive integer \( n \),
\[ |f_{n+1}(x) - f_n(x)| \leq M \cdot \frac{|x-a|^n}{1 \cdot 2 \cdot \ldots \cdot n} \]

Aside:
\[ \int_a^b |g| \leq \int_a^b |g| \leq (\sup_{x \in [a,b]} |g(x)|)(b-a) \]

4. If \( c \) is a positive number, there exists a positive integer \( N \) such that, if \( x \) is in \([A, B] \), then \( f_m(x) \) differs from \( f_n(x) \) by less than \( c \) if \( m > N \) and \( n > N \).

\[ |f_m(x) - f_n(x)| < c \]

5. There exists a function \( f \) with initial set the set of all numbers such that, if \([A, B] \) is an interval and \( c > 0 \), there exists a positive integer \( N \) such that for \( x \) is in \([A, B] \) and \( n > N \), then
\[ |f(x) - f_n(x)| < c \]
6. \( f \) is continuous at each of its points.

7. If, for each positive integer \( n \), \( \varepsilon_n \) is the function defined by \( f_n - f = \varepsilon_n \), so that \( f_n = f + \varepsilon_n \) and if \( \varepsilon_n \) is continuous, then if \( x \) is a number

\[
f(x) - \varepsilon(b + \int_a^x f(s) \, ds) = \int_a^x \varepsilon_{n-1}(s) \, ds - \varepsilon_n(x)
\]

The assumption that there is a number \( x \) such that

\[
|f(x) - \varepsilon(b + \int_a^x f(s) \, ds)| < \delta
\]

is a positive number \( \delta \) leads to a contradiction so that, for every number \( x \)

\[
f(x) = b + \int_a^x f(s) \, ds
\]