\[ \sum_{p} A_p (s-w)^p \]

\[ s_n = \frac{1}{A_n} y_n \]

1. final set of \( s_n \) is not bounded

Suppose \( S \) converges at \( z \neq w \)

\( x : 0 < |x-w| < |z-w| \)

\( S \) converges absolutely at \( z \)

\( \exists \epsilon > 1 \) \( \exists n \) pos. integer such that \( |A_n| > \frac{\epsilon}{|x-w|} \)

\( |A_n||x-w| > \epsilon \)

\( |A_n||x-w|^n > \epsilon^n > \epsilon > 1 \)

Contradiction

\( S \) is totally divergent.

2. \( z \) a point \( \neq w \)

\[ \lim_{n \to \infty} s_n = 0 \Rightarrow \lim_{n \to \infty} |s_n| |z-w| = 0 \]

\[ \Rightarrow \lim_{n \to \infty} |A_n| |z-w|^n = 0 \]
By comparison with geometric series, the power series must converge at \( \frac{1}{2} \). Totally convergent.

3) \( S \) has radius of convergence \( r \) if the final set of \( S \) is bounded, \( S \) does not have limit 0, and \( \frac{1}{r} \) is the greatest number which is a cluster point of \( S \).

\[
\sum_{n} |A_{n}| |z-w|^{n} = \sum_{n} (|A_{n}|^{r} |z-w|^{n})
\]

\[0 < |z-w| < \frac{1}{r}\]

\[
\frac{1}{r} < \frac{1}{|z-w|}
\]

\( m \) pos int such that if \( n \) pos int.

Then \( |A_{m+n}|^{\frac{1}{m+n}} < \frac{1}{|z-w|} \)

\[
0 \leq z \leq \frac{1}{2} \quad \beta \quad \frac{1}{|z-w|}
\]

\( \infty \)

\( |A_{m+n}|^{\frac{1}{m+n}} |z-w| < 1 \)
Actually, \[ |A_{m+n}|^{\frac{1}{m+n}} < \beta = \frac{1-\varepsilon}{|z-w|} \]

\[ |A_{m+n}|^{\frac{1}{m+n}} |z-w| < 1-\varepsilon \]

So again, comparing with the geometric series, we get that power series \( S \) converges at \( z \) where \( 0 < |z-w| < D \).

**Annulus Theorem**

Suppose \( 0 < m < M \), \( w \) is a point, and the point function \( f \) is analytic in the annulus \( A \) to which \( z \) belongs only in case \( z \) is a point \( z \) and \( m < |z-w| < M \).
(1) If \( m < r < R < M \), then

\[
\sum_{j=1}^{4} \int_{R_j} f = 0
\]
(2) If $D_1$ is the disc to which $z$ belongs only in case $|z - w| < \frac{1}{M}$ and $D_2$ is the disc to which $z$ belongs only in case $|z - w| < M$ then there is only one ordered pair $\theta, h$ such that $g$ is an analytic function on $D_1$, $h$ is an analytic function on $D_2$, $g(w) = 0$, and if $z$ belongs to $A$ then
\[ f(z) = g \left( w + \frac{1}{z - w} \right) + h(z). \]

(3) There is a sequence $\{b_n\}_{n=-\infty}^{\infty}$ such that if $m < r < M$ and $n$ is an integer, then
\[ b_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-w)^{r+n+1}} \, dz. \]
moreover, if \( f, g, h \) is the pair of functions from (2), then

\[
b_0 = h(\omega) \quad \text{and for each pos. int. } n,
\]

\[
b_n = \frac{g^{(n)}(\omega)}{n!}, \quad b_n = \frac{h^{(n)}(\omega)}{n!}
\]

Example: if \( f \) analytic

\[
f(z) = f(\omega) + f'(\omega)(z-\omega) + \frac{f''(\omega)}{2!}(z-\omega)^2 + \cdots
\]

\[
S = \sum_{p=0}^{\infty} A_p (z-\omega)^p
\]

\[
\sum_{p=-\infty}^{\infty} A_p (z-\omega)^p
\]

Laurent Series

\[
f(z) = \frac{1}{z-\omega}
\]

\[
f(z) = \frac{1}{(z-\omega)^3} + \frac{1}{z-\omega} + (z-\omega)^4
\]
Can we "reuse" our proof of the Cauchy Integral Formula to get the decomposition?